

## Non-Arbitrage under a Class of Honest Times

Anna Aksamit · Tahir Choulli · Jun  
Deng · Monique Jeanblanc

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**Abstract** This paper quantifies the interplay between the non-arbitrage notion of No-Unbounded-Profit-with-Bounded-Risk (NUPBR hereafter) and additional information generated by a random time. This study complements the one of Aksamit/Choulli/Deng/Jeanblanc [1] in which the authors studied similar topics for the case of stopping at the random time instead, while herein we are concerned with the part after the occurrence of the random time. Given that all the literature —up to our knowledge— proves that the NUPBR notion is always violated after honest times that avoid stopping times in a continuous filtration, herein we propose a *new class of honest times* for which the NUPBR notion can be preserved for some models. For this family of honest times, we elaborate two principal results. The first main result characterizes the pairs of initial market and honest time for which the resulting model preserves the NUPBR property, while the second main result characterizes the honest times that preserve the NUPBR property for any quasi-left continuous model. Furthermore, we construct explicitly “the-after- $\tau$ ” local martingale deflators for a large class of initial models (i.e. models in the small filtration) that are already risk-neutralized.

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Anna Aksamit  
Mathematical Institute, University of Oxford,  
Oxford, United Kingdom  
Monique Jeanblanc  
Laboratoire de Mathématiques et Modélisation d'Évry (LaMME), Université d'Évry Val  
d'Essonne, UMR CNRS 8071, France  
Tahir Choulli (corresponding author)  
Mathematical and Statistical Sciences Depart., University of Alberta, Edmonton, Canada  
E-mail: tchoulli@ualberta.ca  
Jun Deng  
School of Banking and Finance,  
University of International Business and Economics, Beijing, China

## 1 Introduction

This paper complements the study undertaken in [1] about quantifying the exact interplay between an extra information/uncertainty and arbitrage for quasi-left-continuous models<sup>1</sup>. Similarly as in [1], we focus on the non-arbitrage concept of No-Unbounded-Profit-with-Bounded-Risk (NUPBR hereafter), and the extra information is the time of the occurrence of a random time, when it occurs. It is clearly stated in [9] (see [1]) that when the NUPBR is violated, none of the existing method for pricing and optimisation problems works. **Throughout the paper, arbitrages means Unbounded-Profit-with-Bounded-Risk strategies.**

### 1.1 What are the Main Goals and the Related Literature?

Throughout the paper, we consider given a stochastic basis  $(\Omega, \mathcal{G}, \mathbb{F}, P)$ , where  $\mathcal{F}_\infty := \bigvee_{t \geq 0} \mathcal{F}_t \subseteq \mathcal{G}$ , and the filtration  $\mathbb{F} := (\mathcal{F}_t)_{t \geq 0}$  satisfies the usual hypotheses (i.e. right continuity and completeness) and models the flow of “public” information that all agents receive through time. The initial financial market is defined on this basis and is represented by a  $d$ -dimensional semimartingale  $S$  and a riskless asset, with null interest rate. In addition to this initial model, we consider a fixed random time (a non-negative random variable) denoted by  $\tau$ . This random time can represent the death time of an insurer, the default time of a firm, or any occurrence time of an influential event that can impact the market somehow. In this setting, our aim lies in answering the following.

If  $(\Omega, \mathbb{F}, S)$  is arbitrage-free, then what can be said about  $(\Omega, \mathbb{F}, S, \tau)$ ?

After modeling the *new informational system*, this question translates into whether  $(\Omega, \mathbb{G}, S)$  has arbitrages or not. Here  $\mathbb{G}$ , that will be specified mathematically in the next section, is the new flow of information that incorporates the flow  $\mathbb{F}$  and  $\tau$ , as soon as it occurs, and makes  $\tau$  a  $\mathbb{G}$ -stopping time. Thanks to [21] (see also [7] for the continuous case and [20] for the one dimensional case), one can easily prove that  $(\Omega, \mathbb{G}, S)$  satisfies the NUPBR condition if and only if both models  $(\Omega, \mathbb{G}, S^\tau)$  and  $(\Omega, \mathbb{G}, S - S^\tau)$  fulfill the NUPBR condition. In [1], the authors focused on  $(\Omega, \mathbb{G}, S^\tau)$ , while the second part  $(\Omega, \mathbb{G}, S - S^\tau)$  constitutes the main objective of this paper. As it will be mathematically specified later, the NUPBR notion consists, roughly speaking, of “controlling” in some sense the gain processes that are bounded uniformly in time and randomness from below by one. Mathematically speaking, these processes are stochastic integrals with respect to the asset’s price process. Thus, due to the Dellacherie-Mokobodski criterion, the first obstacle in investigating the NUPBR for  $(\Omega, \mathbb{G}, S - S^\tau)$ , lies in whether this model is an integrator for “admissible” but complex (not only buy-and-hold) financial strategies or not.

<sup>1</sup> A quasi-left-continuous model/process is a process that does not jump on predictable stopping times

This boils down to the model fulfilling the semimartingale property (see Theorem 80 in [11] page 401). Thus, the “honest” assumption on  $\tau$  guarantees the preservation of the semimartingale property after  $\tau$ . It is known that (see [19, Théorème 4.14]), in contrast to  $(\mathbb{G}, S^\tau)$ , the semimartingale structures might fail for  $(\mathbb{G}, S - S^\tau)$  when  $\tau$  is arbitrary general. Therefore, for the rest of the paper,  $\tau$  is assumed to be honest, a fact that will be mathematically defined in the next section.

Recently, in [16] and [14], it is proved when honest times avoid stopping times and the filtration is Brownian that the NUPBR property fails for  $(S - S^\tau, \mathbb{G})$ . Thus, our first goal is to answer the following

Is there any  $\tau$  for which NUPBR is preserved for some markets? (1.1)

We answer this question positively, and we focus afterwards on quantifying the interplay between  $\tau$  and the initial market model that is responsible for arbitrages after  $\tau$ . This can be achieved, in our view, by finding a functional  $\mathcal{K}$ , that can be observed using the public information only, such that

$(\mathcal{K}(S), \mathbb{F})$  is arbitrage-free if and only if  $(S - S^\tau, \mathbb{G})$  does. (1.2)

## 1.2 Our Financial and Mathematical Achievements

Our first original contribution proposes a **new class of honest times** for which there are markets that preserve the NUPBR condition after  $\tau$ , and hence our first aim described in (1.1) is reached. Our family of honest times includes all the  $\mathbb{F}$ -stopping times as well as many examples of non  $\mathbb{F}$ -stopping times. By considering this subclass of honest times throughout the paper, our principal novelty resides in achieving our second aim of (1.2), and describe as explicit as possible the functional  $\mathcal{K}$ . As a result, honest times belonging to our class might induce “the-after- $\tau$ ” arbitrages only if the initial market jumps.

This paper is organized as follows. In the following section (Section 2), we present our main results, their immediate consequences, and/or their economic and financial interpretations. In this section, we also develop many examples and show how the main ideas came into play. Section 3 deals with the derivation of explicit local martingale deflators for a class of processes. The last section (Section 4) focuses on proving the main theorem and other related results announced. The paper contains also an appendix where some of the existing and/or new technical results are summarized.

## 2 The Main Results and their Financial Interpretations

This section contains three subsections. The first subsection defines notations and the NUPBR concept, while the second subsection develops simple examples of informational markets and explains how some ingredients of the main

results play natural and important rôles. The last subsection announces the principal results, their applications, and gives their financial meanings as well.

## 2.1 Notations and Preliminaries

In what follows,  $\mathbb{H}$  denotes a filtration satisfying the usual hypotheses. The set of  $\mathbb{H}$ -martingales is denoted by  $\mathcal{M}(\mathbb{H})$ . As usual,  $\mathcal{A}^+(\mathbb{H})$  denotes the set of increasing, right-continuous,  $\mathbb{H}$ -adapted and integrable processes.

If  $\mathcal{C}(\mathbb{H})$  is a class of  $\mathbb{H}$ -adapted processes, we denote by  $\mathcal{C}_0(\mathbb{H})$  the set of processes  $X \in \mathcal{C}(\mathbb{H})$  with  $X_0 = 0$ , and by  $\mathcal{C}_{loc}(\mathbb{H})$  the set of processes  $X$  such that there exists a sequence  $(T_n)_{n \geq 1}$  of  $\mathbb{H}$ -stopping times that increases to  $+\infty$  and the stopped processes  $X^{T_n}$  belong to  $\mathcal{C}(\mathbb{H})$ . We put  $\mathcal{C}_{0,loc} = \mathcal{C}_0 \cap \mathcal{C}_{loc}$ .

For a process  $K$  with  $\mathbb{H}$ -locally integrable variation, we denote by  $K^{o,\mathbb{H}}$  its dual optional projection. The dual predictable projection of  $K$  is denoted  $K^{p,\mathbb{H}}$ . For a process  $X$ , we denote  ${}^{o,\mathbb{H}}X$  (resp.  ${}^{p,\mathbb{H}}X$ ) its optional (resp. predictable) projection with respect to  $\mathbb{H}$ .

For a finite-dimensional  $\mathbb{H}$ -semimartingale  $X$ , the set  $\mathcal{L}(X, \mathbb{H})$  is the set of  $\mathbb{H}$ -predictable processes having the same dimension as  $X$  and being integrable w.r.t.  $X$  and for  $H \in \mathcal{L}(X, \mathbb{H})$ , the resulting integral is the one-dimensional process denoted by  $H \cdot X_t := \int_0^t H_s dX_s$ . **Throughout the paper, stochastic processes have arbitrary finite dimension** (in case it is not specified). We recall the notion of non-arbitrage that is addressed in this paper.

**Definition 2.1** An  $\mathbb{H}$ -semimartingale  $X$  satisfies the *No-Unbounded-Profit-with-Bounded-Risk* condition under  $(\mathbb{H}, Q)$  if for any  $T' \in (0, +\infty)$ , the set

$$\mathcal{K}_{T'}(X) := \left\{ (H \cdot X)_{T'} \mid H \in \mathcal{L}(X, \mathbb{H}), \text{ and } H \cdot X \geq -1 \right\}$$

is bounded in probability under  $Q$ . Often, we abbreviate by saying that  $X$  satisfies the NUPBR $(\mathbb{H}, Q)$ , or the model  $(X, \mathbb{H}, Q)$  satisfies the NUPBR. When  $Q \sim P$ , we simply drop the probability for short and simplifying the notations.

For more details about this non-arbitrage condition and its relationship to the literature, we refer the reader to Aksamit et al. [1]. The NUPBR property is intimately related to the existence of a  $\sigma$ -martingale density. Below, we recall the definition of  $\sigma$ -martingale and  $\sigma$ -martingale density for a process.

**Definition 2.2** An  $\mathbb{H}$ -adapted process  $X$  is called an  $(\mathbb{H}, \sigma)$ -martingale if there exists a real-valued  $\mathbb{H}$ -predictable process  $\phi$  such that

$$0 < \phi \leq 1, \quad \text{and } \phi \cdot X \text{ is an } \mathbb{H}\text{-martingale.}$$

If  $X$  is  $\mathbb{H}$ -adapted, we call  $(\mathbb{H}, \sigma)$ -martingale density for  $X$  (**also called  $\mathbb{H}$ -deflator for  $X$** ), any real-valued positive  $\mathbb{H}$ -local martingale  $L$  such that  $XL$  is an  $(\mathbb{H}, \sigma)$ -martingale. The set of all  $\mathbb{H}$ -deflators for  $X$  is denoted by

$$\mathcal{L}_\sigma(X, \mathbb{H}) := \left\{ L \in \mathcal{M}_{loc}(\mathbb{H}) \mid L > 0, \quad LX \text{ is an } (\mathbb{H}, \sigma)\text{-martingale} \right\}. \quad (2.1)$$

The equivalence between  $\text{NUPBR}(\mathbb{H})$  for a process  $X$  and  $\mathcal{L}_\sigma(X, \mathbb{H}) \neq \emptyset$  is established in [1] (see Proposition 2.3) when the horizon may be infinite, and in [21] for the case of finite horizon.

Beside the initial model  $(\Omega, \mathbb{F}, P, S)$  in which  $S$  is assumed to be a **quasi-left-continuous semimartingale**, we consider a random time  $\tau$ , to which we associate the process  $D$  and the filtration  $\mathbb{G}$  given by

$$D := I_{[\tau, +\infty[}, \quad \mathbb{G} = (\mathcal{G}_t)_{t \geq 0}, \quad \mathcal{G}_t := \bigcap_{s > t} \left( \mathcal{F}_s \vee \sigma(D_u, u \leq s) \right).$$

The filtration  $\mathbb{G}$  is the smallest right-continuous filtration which contains  $\mathbb{F}$  and makes  $\tau$  a stopping time. In the probabilistic literature,  $\mathbb{G}$  is called the progressive enlargement of  $\mathbb{F}$  with  $\tau$ . In addition to  $\mathbb{G}$  and  $D$ , we associate to  $\tau$  two important  $\mathbb{F}$ -supermartingales: the  $\mathbb{F}$ -optional projection of  $I_{[0, \tau]}$  denoted  $Z$ , and the  $\mathbb{F}$ -optional projection of  $I_{[0, \tau]}$ , denoted  $\tilde{Z}$ , which satisfy

$$Z_t := P(\tau > t \mid \mathcal{F}_t) \text{ and } \tilde{Z}_t := P(\tau \geq t \mid \mathcal{F}_t). \quad (2.2)$$

$Z$  is right-continuous with left limits, while  $\tilde{Z}$  admits right limits and left limits. An important  $\mathbb{F}$ -martingale, denoted by  $m$ , is given by

$$m := Z + D^{o, \mathbb{F}}, \quad (2.3)$$

where  $D^{o, \mathbb{F}}$  is the  $\mathbb{F}$ -dual optional projection of  $D = I_{[\tau, \infty[}$  (Note that  $Z$  is bounded and  $D^{o, \mathbb{F}}$  is nondecreasing and integrable).

To distinguish the effect of filtration, we will denote  $\langle \cdot, \cdot \rangle^{\mathbb{F}}$ , or  $\langle \cdot, \cdot \rangle^{\mathbb{G}}$  to specify the sharp bracket (predictable covariation process) calculated in the filtration  $\mathbb{F}$  or  $\mathbb{G}$ , if confusion may rise. We recall that, for general semimartingales  $X$  and  $Y$ , the sharp bracket is (if it exists) the dual predictable projection of the covariation process  $[X, Y]$ . For the reader's convenience, we recall the definition of honest time.

**Definition 2.3** A random time  $\sigma$  is honest, if for any  $t$ , there exists an  $\mathcal{F}_t$  measurable r.v.  $\sigma_t$  such that  $\sigma I_{\{\sigma < t\}} = \sigma_t I_{\{\sigma < t\}}$ .

We refer to Jeulin [19, Chapter 5] and Barlow [6] for more information about honest times. In this paper, we restrict our study to the following subclass  $\mathcal{H}$  of random times:

$$\mathcal{H} := \{ \tau \text{ is an honest time satisfying } Z_\tau I_{\{\tau < +\infty\}} < 1, \quad P - a.s. \} \quad (2.4)$$

*Remark 2.4* 1) It is clear that any  $\mathbb{F}$ -stopping time belongs to  $\mathcal{H}$  (we even have  $Z_\tau I_{\{\tau < +\infty\}} = 0$ ), and hence our subclass of honest times is not empty.  
 2) In the case where  $\mathbb{F}$  is the completed Brownian filtration, we consider the following  $\mathbb{F}$ -stopping times

$$U_0^\epsilon = V_0^\epsilon = 0, \quad U_n^\epsilon := \inf\{t \geq V_{n-1}^\epsilon : B_t = \epsilon\}, \quad V_n^\epsilon := \inf\{t \geq U_n^\epsilon : B_t = 0\},$$

where  $\epsilon \in (0, 1)$  and  $B$  is a one dimensional standard Brownian motion. Then,

$$\tau := \sup\{V_n^\epsilon : V_n^\epsilon \leq T_1\},$$

where  $T_1 := \inf\{t \geq 0 : B_t = 1\}$ , is an honest time which is not a stopping time, and belongs to  $\mathcal{H}$  (see [3] for detailed proof). Other examples of elements of  $\mathcal{H}$  that are not stopping times are given in the next subsection.

We conclude this subsection with the following lemma, obtained in [1].

**Lemma 2.5** *Let  $X$  be an  $\mathbb{H}$ -predictable process with finite variation. Then  $X$  satisfies  $NUPBR(\mathbb{H})$  if and only if  $X \equiv X_0$  (i.e. the process  $X$  is constant).*

## 2.2 Particular Cases and Examples

In this subsection, by analysing particular cases and examples, we obtain some results vital for understanding the exact interplay between the features of the initial markets and the honest time under consideration. The following simple lemma plays a key role in this analysis.

**Lemma 2.6** *The following assertions hold.*

(a) *Let  $M$  be an  $\mathbb{F}$ -local martingale, and  $\tau$  be an honest time. Then the process  $\widehat{M}$ , defined as*

$$\widehat{M} := M - M^\tau + (1 - Z_-)^{-1} I_{\llbracket \tau, +\infty \llbracket} \cdot \langle M, m \rangle^\mathbb{F}, \quad (2.5)$$

*is a  $\mathbb{G}$ -local martingale.*

(b) *If  $\tau \in \mathcal{H}$ , then the  $\mathbb{G}$ -predictable process  $(1 - Z_-)^{-1} I_{\llbracket \tau, +\infty \llbracket}$  is  $\mathbb{G}$ -locally bounded.*

*Proof 1)* Assertion (a) is a standard result on progressive enlargement of filtration with honest times (see [6, 12, 19]).

**2)** Herein we prove assertion (b). It is known [12, Chapter XX] that  $Z = \widetilde{Z}$  on  $\llbracket \tau, +\infty \llbracket$ , and

$$\llbracket \tau, +\infty \llbracket \subset \{Z_- < 1\} \cap \{\widetilde{Z} < 1\} \subset \{Z_- < 1\} \cap \{Z < 1\}.$$

Then, since  $\tau \in \mathcal{H}$ , we deduce that  $\llbracket \tau, +\infty \llbracket \subset \{Z < 1\}$ , and hence the process

$$X := (1 - Z)^{-1} I_{\llbracket \tau, +\infty \llbracket},$$

is càdlàg  $\mathbb{G}$ -adapted with values in  $[0, +\infty)$  (finite values). Combining these with  $\llbracket \tau, +\infty \rrbracket \subset \{Z_- < 1\}$ , we can prove easily that

$$T_n := \inf\{t \geq 0 : X_t \geq n\} \uparrow +\infty \text{ and } \max(X^{T_n-}, X_-^{T_n}) \leq n, \quad P - a.s..$$

Thus,  $X_- = (1 - Z_-)^{-1} I_{\llbracket \tau, +\infty \rrbracket}$  is locally bounded, and the proof of the lemma is completed.  $\square$

**Theorem 2.7** *Suppose that  $\tau \in \mathcal{H}$ . If  $S$  is continuous and satisfies  $\text{NUPBR}(\mathbb{F})$ , then  $S - S^\tau$  satisfies  $\text{NUPBR}(\mathbb{G})$ .*

*Remark 2.8* This theorem follows from one of our principal results stated in the next subsection. However, due to the simplicity of its proof that does not require any further technicalities, we opted for detailing this proof below.

*Proof of Theorem 2.7:* Let  $S = (S^1, \dots, S^d)$  be a  $d$ -dimensional continuous process satisfying  $\text{NUPBR}(\mathbb{F})$ . Then, there exists a positive  $\mathbb{F}$ -local martingale  $L$  such that  $LS$  is an  $(\mathbb{F}, \sigma)$ -martingale. Since  $S$  is continuous and  $L$  is a local martingale, we deduce that  $\sup_{u \leq \cdot} |S_u| \sup_{u \leq \cdot} |\Delta L_u|$  is locally integrable. Thus, thanks to Proposition 3.3 in [4] and  $\sum_{i=1}^d \Delta(LS^i) = \sum_{i=1}^d S^i \Delta L \geq -d \sup_{u \leq \cdot} |S_u| \sup_{u \leq \cdot} |\Delta L_u|$ , we conclude that  $LS$  is an  $\mathbb{F}$ -local martingale. Consider a sequence of  $\mathbb{F}$ -stopping times  $(T_n)_{n \geq 1}$  that increases to infinity such that both  $L^{T_n}$  and  $L^{T_n} S^{T_n}$  are martingales, and put  $Q_n := (L_{T_n}/L_0) \cdot P \sim P$ . Then,  $S^{(n)} := S^{T_n}$  is an  $(\mathbb{F}, Q_n)$ -martingale on the one hand. On the other hand, in virtue of Proposition A.2,  $S - S^\tau$  satisfies  $\text{NUPBR}(\mathbb{G})$  if and only if  $S^{(n)} - (S^{(n)})^\tau$  satisfies  $\text{NUPBR}(\mathbb{G})$  under  $Q_n$ , for all  $n \geq 1$ . This shows that, without loss of generality, one need to prove the theorem only when  $S$  is an  $\mathbb{F}$ -martingale. Thus, for the rest of the proof, we assume that  $S$  is an  $\mathbb{F}$ -martingale. Thanks to Lemma 2.6, the process  $Y^\mathbb{G} := \mathcal{E}((1 - Z_-)^{-1} I_{\llbracket \tau, +\infty \rrbracket} \cdot \widehat{m}^c)$  is a well defined continuous real-valued and positive  $\mathbb{G}$ -local martingale, where  $m^c$  is the continuous  $\mathbb{F}$ -local martingale part of  $m$ , and  $\widehat{m}^c$  is defined as in (2.5). Thanks to the continuity of  $S$  and (2.5), we get

$$\begin{aligned} S - S^\tau + \left[ S - S^\tau, \frac{I_{\llbracket \tau, +\infty \rrbracket}}{1 - Z_-} \cdot \widehat{m}^c \right] &= S - S^\tau + (1 - Z_-)^{-1} I_{\llbracket \tau, +\infty \rrbracket} \cdot \langle S, m \rangle^\mathbb{F} \\ &= \widehat{S} \in \mathcal{M}_{loc}(\mathbb{G}). \end{aligned}$$

Therefore, a combination of this and Itô's formula applied to  $(S - S^\tau)Y^\mathbb{G}$ , we conclude that this latter process is a  $\mathbb{G}$ -local martingale. This proves the  $\text{NUPBR}(\mathbb{G})$  for  $S - S^\tau$ , and the proof of the theorem is achieved.  $\square$

*Remark 2.9* Theorem 2.7 asserts clearly that, if  $\tau \in \mathcal{H}$ , the jumps of  $S$  have significant impact on  $\mathbb{G}$ -arbitrages for  $S - S^\tau$ . Thus, the following natural question arises:

$$\text{Does the condition } \{\Delta S \neq 0\} \cap \llbracket \tau \rrbracket = \emptyset \text{ impact } \mathbb{G}\text{-arbitrages?} \quad (2.6)$$

*Example 2.10* Suppose that  $\mathbb{F}$  is generated by a Poisson process  $N$  with intensity one. Consider two real numbers  $a > 0$  and  $\mu > 1$ , and set

$$\tau := \sup\{t \geq 0 : Y_t := \mu t - N_t \leq a\}, \quad M_t := N_t - t. \quad (2.7)$$

It can be proved easily, see [3], that  $\tau \in \mathcal{H}$  is finite almost surely, and the associated processes  $Z$  and  $\tilde{Z}$  are given by

$$Z = \Psi(Y - a)I_{\{Y \geq a\}} + I_{\{Y < a\}} \quad \text{and} \quad \tilde{Z} = \Psi(Y - a)I_{\{Y > a\}} + I_{\{Y \leq a\}}.$$

Here  $\Psi(u) := P(\sup_{t \geq 0} Y_t > u)$  is the ruin probability associated to the process  $Y$  (see [5]). As a result we have

$$1 - Z_- = [1 - \Psi(Y_- - a)] I_{\{Y_- > a\}}, \quad (2.8)$$

and we can prove that

$$m = m_0 + \phi \bullet M, \quad \text{where} \quad (2.9)$$

$$\phi := [\Psi(Y_- - a - 1) - \Psi(Y_- - a)] I_{\{Y_- > 1+a\}} + [1 - \Psi(Y_- - a)] I_{\{a < Y_- \leq 1+a\}}.$$

Suppose that  $S = I_{\{a \leq Y_- < a+1\}} \bullet M$ . Then, in virtue of Lemma 2.5, the process  $S - S^\tau$  (which is not null) violates NUPBR( $\mathbb{G}$ ) if it is  $\mathbb{G}$ -predictable with finite variation. This latter fact is equivalent to  $\hat{S}$  ( $\mathbb{G}$ -local martingale part of  $S - S^\tau$ ) being null, or equivalently  $\langle \hat{S}, \hat{S} \rangle^\mathbb{G} \equiv 0$ . By using Lemma 2.6 and Itô's lemma and putting  $V_t = t$ , we derive

$$\begin{aligned} [\hat{S}, \hat{S}] &= I_{\tau, +\infty} \bullet [S] = I_{\tau, +\infty} \bullet S + I_{\{a < Y_- \leq a+1\}} I_{\tau, +\infty} \bullet V \\ &= I_{\tau, +\infty} \bullet \hat{S} + I_{\{a < Y_- \leq a+1\}} I_{\tau, +\infty} \left( 1 - \frac{\phi}{1 - Z_-} \right) \bullet V, \quad (2.10) \\ &= I_{\tau, +\infty} \bullet \hat{S} \quad \text{is a } \mathbb{G}\text{-local martingale.} \end{aligned}$$

The last equality is due to  $\phi \equiv 1 - Z_-$  on  $\{a \leq Y_- < a+1\} \cap \tau, +\infty$ . This proves that  $\hat{S} \equiv 0$ , and hence  $S - S^\tau$  violates NUPBR( $\mathbb{G}$ ).

*Example 2.11* Consider the same setting and notations as Example 2.10, except for the initial market model that we suppose having the form of  $S = I_{\{Y_- > a+1\}} \bullet M$  instead. Then, by combining Lemma 2.6, Itô's lemma and similar calculation as in (2.10), we deduce that both  $Y^\mathbb{G} := \mathcal{E}(\xi \bullet \hat{S})$  and  $Y^\mathbb{G}(S - S^\tau)$  are  $\mathbb{G}$ -local martingales and  $Y^\mathbb{G} > 0$ . Here  $\xi$  is given by

$$\xi := \frac{\Psi(Y_- - a - 1) - 1}{2 - \Psi(Y_- - a) - \Psi(Y_- - a - 1)} I_{\{Y_- > a+1\}} I_{\tau, +\infty} \bullet.$$

This proves that  $S - S^\tau$  satisfies NUPBR( $\mathbb{G}$ ).



*Remark 2.12* 1) The economics/financial meaning of Examples 2.10 and 2.11 resides in the following: The random time defined in (2.7) represents the last time the cash reserve of a firm does not exceed the level  $a$ . Then, in Example 2.10 (respectively in Example 2.11) one can consider any security whose price process lives on  $\{a \leq Y_- < 1 + a\}$  (respectively on  $\{Y_- > 1 + a\}$ ).

2) Remark that, in both Examples 2.10 and 2.11, the graph of the random time  $\tau$  is included in a union of countable graphs of predictable stopping times. Hence, due to the quasi-left-continuity of  $S$ , we immediately conclude that  $\{\Delta S \neq 0\} \cap \llbracket \tau \rrbracket$  is empty for both examples. This answers negatively (2.6).

### 2.3 Main Results and Their Applications

Our first main result requires the following easy and interesting lemma.

**Lemma 2.13** *Suppose that  $\tau \in \mathcal{H}$  and is finite almost surely. Then,*

$$V^{\mathbb{F}} := \sum I_{\{\tilde{Z}=1 > Z_-\}}, \quad (2.11)$$

*is càdlàg with finite values, and hence is  $\mathbb{F}$ -locally integrable.*

*Proof* Thanks to Proposition B.1-(b), there exists a sequence of  $\mathbb{F}$ -stopping times,  $(\sigma_n)_{n \geq 1}$  that increases to infinity almost surely and  $1 \leq n^2(1 - Z_{t-})^2$  on  $\{Z_{t-} < 1\} \cap \{t \leq \sigma_n\}$ . Thus, for any nonnegative and bounded  $\mathbb{F}$ -optional process  $H$ , we have

$$\begin{aligned} (H \cdot V^{\mathbb{F}})^{\sigma_n} &\leq n^2 \left( \sum H(1 - Z_-)^2 I_{\{\tilde{Z}=1 > Z_-\}} \right)^{\sigma_n} \\ &= n^2 \left( \sum H(\Delta m)^2 I_{\{\tilde{Z}=1 > Z_-\}} \right)^{\sigma_n} \leq n(H \cdot [m, m])^{\sigma_n}. \end{aligned}$$

Therefore, the proof of the lemma follows immediately from combining the above inequality and the fact that  $[m, m] \in \mathcal{A}_{loc}^+(\mathbb{F})$ .  $\square$

In the following, we announce our first main result.

**Theorem 2.14** *Suppose that  $S$  is an  $\mathbb{F}$ -quasi-left-continuous semimartingale, and  $\tau \in \mathcal{H}$  is finite almost surely. Then, the following are equivalent.*

- (a)  $S - S^\tau$  satisfies NUPBR( $\mathbb{G}$ ).
- (b)  $(1 - Z_-) \cdot \mathcal{T}_a(S)$  satisfies NUPBR( $\mathbb{F}$ ).
- (c)  $I_{\{Z_- < 1\}} \cdot \mathcal{T}_a(S)$  satisfies NUPBR( $\mathbb{F}$ ), where

$$\mathcal{T}_a(S) := S - [S, V^{\mathbb{F}}] = S - \sum \Delta S I_{\{\tilde{Z}=1 > Z_-\}}. \quad (2.12)$$

The proof of (a) $\implies$ (b) is technical and requires notations. Thus, for the reader's convenience, we postponed the whole theorem's proof to Section 4.

*Remark 2.15* (a) The theorem asserts, in a precise and deep manner, that  $\mathbb{G}$ -arbitrages for the process  $S - S^\tau$  are intimately related to the interplay between the jumps of  $S$  and the jumps of  $\tilde{Z}$  to the value one.

(b) Theorem 2.14 claims that  $S - S^\tau$  is arbitrage-free under  $\mathbb{G}$  if and only if the part  $\mathcal{T}_a(S)$  (of  $S$ ) is arbitrage-free under  $\mathbb{F}$  on the set  $\{Z_- < 1\}$ . As a result, this allows us to single out practical cases for which the NUPBR is preserved after  $\tau$ , as outlined in the forthcoming Corollary 2.16 and Theorem 2.18.

**Corollary 2.16** *Suppose that  $S$  is  $\mathbb{F}$ -quasi-left-continuous, and  $\tau \in \mathcal{H}$  is finite almost surely. Then the following assertions hold:*

(a) *If  $\left(S, \sum(\Delta S)I_{\{\tilde{Z}=1>Z_-\}}\right)$  satisfies NUPBR( $\mathbb{F}$ ), then  $S - S^\tau$  satisfies NUPBR( $\mathbb{G}$ ).*

(b) *If  $S$  satisfies NUPBR( $\mathbb{F}$ ) and  $\{\Delta S \neq 0\} \cap \{\tilde{Z} = 1 > Z_-\} = \emptyset$ , then  $S - S^\tau$  satisfies NUPBR( $\mathbb{G}$ ).*

*Remark 2.17* 1) Assertion (b) asserts that if  $S$  does not jump on  $\{\tilde{Z} = 1 > Z_-\}$ , then no arbitrage under  $\mathbb{G}$  will occur in the part “after- $\tau$ ”. Assertion (a) gives much weaker assumption than assertion (b), as it assumes that  $L\left(\sum \Delta SI_{\{\tilde{Z}=1>Z_-\}}\right) \in \mathcal{M}_{loc}(\mathbb{F})$  for some  $L \in \mathcal{L}_\sigma(S, \mathbb{F})$  (defined in (2.1)), while assertion (b) assumes that  $\sum \Delta SI_{\{\tilde{Z}=1>Z_-\}}$  is null.

*Proof of Corollary 2.16:* It is obvious that assertion (a) follows directly from combining  $(1 - Z_-) \cdot \mathcal{T}_a(S) = (1 - Z_-, -(1 - Z_-)) \cdot \left(S, \sum \Delta SI_{\{\tilde{Z}=1>Z_-\}}\right)$  and Theorem 2.14. Due to  $\{\Delta S \neq 0\} \cap \{\tilde{Z} = 1 > Z_-\} = \emptyset$ , assertion (b) follows from assertion (a), and the proof of the corollary is achieved.  $\square$

In the spirit of further applicability of Theorem 2.14, we state the following

**Theorem 2.18** *Suppose that  $\tau \in \mathcal{H}$ . Let  $\mu$  be the optional random measure associated to the jumps of  $S$ , and  $\nu^\mathbb{F}$  and  $\nu^\mathbb{G}$  be the  $\mathbb{F}$ -compensator and the  $\mathbb{G}$ -compensator of  $\mu$  and  $I_{\llbracket \tau, +\infty \rrbracket} \cdot \mu$  respectively. If  $S$  satisfies NUPBR( $\mathbb{F}$ ) and*

$$I_{\llbracket \tau, +\infty \rrbracket} \cdot \nu^\mathbb{F} \text{ is equivalent to } \nu^\mathbb{G} \quad P - a.s., \quad (2.13)$$

*then  $S - S^\tau$  satisfies NUPBR( $\mathbb{G}$ ).*

The proof of this theorem follows from Theorem 2.14 as long as we prove that, under (2.13),  $S$  satisfies the NUPBR( $\mathbb{F}$ ) if and only if  $\mathcal{T}_a(S)$  satisfies the NUPBR( $\mathbb{F}$ ). This proof is technical, and thus it is delegated to Section 4.

*Remark 2.19* Remark that we always have the absolute continuity  $\nu^\mathbb{G} \ll I_{\llbracket \tau, +\infty \rrbracket} \cdot \nu^\mathbb{F}$   $P - a.s.$  This follows from the fact that  $\nu^\mathbb{G}$  is absolutely continuous with respect to  $\nu^\mathbb{F}$  and it lives on  $\llbracket \tau, +\infty \rrbracket$  only.

(a) **The Lévy Case:** Suppose that  $S$  is a Lévy process and  $F(dx)$  is its Lévy measure under  $\mathbb{F}$ , then  $\nu^\mathbb{F}(dt, dx) = F(dx)dt$  and  $\nu^\mathbb{G}(dt, dx) = I_{\llbracket \tau, +\infty \rrbracket} F_t^\mathbb{G}(dx)dt$ , where  $F_t^\mathbb{G}(dx)$  is its Lévy measure under  $\mathbb{G}$ . Thus, Theorem 2.18 asserts that if  $P \otimes \lambda$  almost every  $(\omega, t)$  ( $\lambda(dt) = dt$ ),  $F_t^\mathbb{G}(\omega, dx) = f(t, x, \omega)F(dx)$  for some

real-valued and positive functional  $f(t, x, \omega)$ , then  $S - S^\tau$  satisfies  $\text{NUPBR}(\mathbb{G})$ . For more practical Lévy cases, we refer the reader to [13].

(b) **Examples 2.10–2.11 versus Theorem 2.18:** In the context of Example 2.10, we easily calculate  $\nu^\mathbb{R}(dt, dx) = I_{\{a < Y_{t-} \leq a+1\}} \delta_1(dx) dt$  and  $\nu^\mathbb{G}(dt, dx) = I_{\tau, +\infty}[\cdot](t) I_{\{a < Y_{t-} \leq a+1\}} (1 - \phi_t / (1 - Z_{t-})) \delta_1(dx) dt \equiv 0$  which is not equivalent to  $I_{\tau, +\infty}[\cdot] \cdot \nu^\mathbb{R}$ . This example shows that (2.13) can be violated. Therefore, in those circumstances, we can not conclude whether  $S - S^\tau$  satisfies  $\text{NUPBR}(\mathbb{G})$  or not directly from Theorem 2.18.

For the case of Example 2.11, we have  $\nu^\mathbb{R}(dt, dx) = I_{\{Y_{t-} > a+1\}} \delta_1(dx) dt$  and  $\nu^\mathbb{G}(dt, dx) = I_{\tau, +\infty}[\cdot](t) I_{\{Y_{t-} > a+1\}} (1 - \phi_t / (1 - Z_{t-})) \delta_1(dx) dt$  which is equivalent to  $I_{\tau, +\infty}[\cdot] \cdot \nu^\mathbb{R}$  since  $\{Y_- > a+1\} \subset \{\phi < 1 - Z_-\}$   $P \otimes dt$ -a.e. Thus, Theorem 2.18 allows us to conclude that  $S - S^\tau$  fulfills the  $\text{NUPBR}(\mathbb{G})$ .

The rest of this subsection describes models of  $\tau$  preserving the  $\text{NUPBR}$ .

**Theorem 2.20** *Assume that  $\tau \in \mathcal{H}$ . Then, the following are equivalent.*

- (a) *The set  $\{\tilde{Z} = 1 > Z_-\}$  is accessible (i.e. it is contained in a countable union of graphs of  $\mathbb{F}$ -predictable stopping times).*
- (b) *For every (bounded)  $\mathbb{F}$ -quasi-left-continuous martingale  $X$ , the process  $X - X^\tau$  satisfies  $\text{NUPBR}(\mathbb{G})$ .*
- (b') *For any probability  $Q \sim P$  and every (bounded)  $\mathbb{F}$ -quasi-left-continuous  $X \in \mathcal{M}(Q, \mathbb{F})$ , the process  $X - X^\tau$  satisfies  $\text{NUPBR}(\mathbb{G})$ .*
- (c) *For every (bounded)  $\mathbb{F}$ -quasi-left-continuous process  $X$  satisfying  $\text{NUPBR}(\mathbb{F})$ , the process  $X - X^\tau$  satisfies  $\text{NUPBR}(\mathbb{G})$ .*

*Proof* The proof of the proposition is organized in three parts, where we prove (a)  $\iff$  (b), (b)  $\iff$  (b') and (b')  $\iff$  (c) respectively.

1) We start by proving that (a)  $\implies$  (b). Suppose that the thin set  $\{\tilde{Z} = 1 > Z_-\}$  is accessible. Then, for any  $\mathbb{F}$ -quasi-left-continuous martingale  $X$ , we have  $\{\Delta X \neq 0\} \cap \{\tilde{Z} = 1 > Z_-\} = \emptyset$ . Hence, thanks to Corollary 2.16-(d), we deduce that  $X - X^\tau$  satisfies  $\text{NUPBR}(\mathbb{G})$ . This completes the proof of (a)  $\implies$  (b). To prove the reverse, assuming that assertion (b) holds, we consider a sequence of stopping times  $(T_n)_{n \geq 1}$  that exhausts the thin set  $\{\tilde{Z} = 1 > Z_-\}$  (i.e.,

$$\{\tilde{Z} = 1 > Z_-\} = \bigcup_{n=1}^{+\infty} \llbracket T_n \rrbracket.$$

Then, each  $T_n$  – that we denote by  $T$  for the sake of simplicity – can be decomposed into a totally inaccessible part  $T^i$  and an accessible part  $T^a$  as  $T = T^i \wedge T^a$ . Consider the following quasi-left-continuous  $\mathbb{F}$ -martingale

$$M := V - V^{p, \mathbb{F}} =: V - \tilde{V},$$

where  $V := I_{\tau, +\infty}[\cdot]$ . Then, since  $\{T^i < +\infty\} \subset \{\tilde{Z}_{T^i} = 1\}$ , we deduce that  $\{T^i < +\infty\} \subset \{\tau \geq T^i\}$  and hence

$$I_{\tau, +\infty}[\cdot] \cdot M = -I_{\tau, +\infty}[\cdot] \cdot \tilde{V} \quad \text{is } \mathbb{G}\text{-predictable.}$$

Then, the finite variation and  $\mathbb{G}$ -predictable process,  $I_{\llbracket \tau, +\infty \rrbracket} \bullet M$ , satisfies  $\text{NUPBR}(\mathbb{G})$  if and only if it is null, or equivalently

$$0 = E \left( I_{\llbracket \tau, +\infty \rrbracket} \bullet \tilde{V}_\infty \right) = E \left( \int_0^\infty (1 - Z_{s-}) d\tilde{V}_s \right) = E \left( (1 - Z_{T^i-}) I_{\{T^i < +\infty\}} \right).$$

Therefore, we conclude that  $T^i = +\infty$ ,  $P$ -a.s., and the stopping time  $T$  is an accessible stopping time. This ends the proof of (a)  $\iff$  (b).

**2)** It is easy to see that the implication (b')  $\implies$  (b) follows from taking  $Q = P$ . To prove the reverse sense, we suppose given  $Q \sim P$  and an  $\mathbb{F}$ -quasi-left-continuous  $X \in \mathcal{M}(\mathbb{F}, Q)$ . Then, put

$$Z_t^\mathbb{F} := E \left( \frac{dQ}{dP} \middle| \mathcal{F}_t \right) =: \mathcal{E}_t(N), \quad Y := \left( \frac{\mathcal{E}(N^{(qc)})X}{\mathcal{E}(N^{(qc)})} \right) \quad \text{and} \quad N^{(qc)} := N - I_{\bigcup_n \llbracket \sigma_n \rrbracket} \bullet N,$$

where  $(\sigma_n)_n$  is the sequence of  $\mathbb{F}$ -predictable stopping times that exhausts all the predictable jumps of  $N$ . In other words,  $N^{(qc)}$  is the  $\mathbb{F}$ -quasi-left-continuous local martingale part of  $N$ . Then, due to the quasi-left-continuity of  $X$ , simple calculations show that  $Y$  is an  $\mathbb{F}$ -quasi-left-continuous martingale. Therefore, by directly applying assertion (b) to  $Y$ , we conclude that  $Y - Y^\tau = \left( \frac{\mathcal{E}(N^{(qc)})(X - X^\tau) + X^\tau(\mathcal{E}(N^{(qc)}) - \mathcal{E}(N^{(qc)})^\tau)}{\mathcal{E}(N^{(qc)}) - \mathcal{E}(N^{(qc)})^\tau} \right)$  satisfies

$\text{NUPBR}(\mathbb{G})$ . This implies the existence of a real-valued positive  $\mathbb{G}$ -local martingale  $Z^\mathbb{G}$  such that both processes  $Z^\mathbb{G} \mathcal{E}(N^{(qc)})(X - X^\tau)$  and  $Z^\mathbb{G} \mathcal{E}(N^{(qc)})$  are  $\sigma$ -martingales under  $(\mathbb{G}, P)$ . Since  $Z^\mathbb{G} \mathcal{E}(N^{(qc)})$  is positive and thanks to Proposition 3.3 and Corollary 3.5 of [4] (which states that a non-negative  $\sigma$ -martingale is a local martingale), we deduce that  $Z^\mathbb{G} \mathcal{E}(N^{(qc)})$  is a real-valued positive element of  $\mathcal{M}_{loc}(\mathbb{G}, P)$  such that  $Z^\mathbb{G} \mathcal{E}(N^{(qc)})(X - X^\tau)$  is a  $\sigma$ -martingale. This proves that  $X - X^\tau$  satisfies  $\text{NUPBR}(\mathbb{G})$ , and the proof of (b)  $\iff$  (b') is completed.

**3)** Remark that (c)  $\implies$  (b') is obvious, and hence we focus on proving the reverse only. Suppose that assertion (b') holds, and consider an  $\mathbb{F}$ -quasi-left-continuous process  $X$  satisfying  $\text{NUPBR}(\mathbb{F})$ . Then, there exists a real-valued and positive  $\mathbb{F}$ -local martingale  $Y$ , and a real-valued and  $\mathbb{F}$ -predictable process  $\phi$  such that

$$0 < \phi \leq 1 \quad Y(\phi \bullet X) \quad \text{is an } \mathbb{F}\text{-martingale.}$$

Let  $(T_n)$  be a sequence of  $\mathbb{F}$ -stopping times that increases to infinity (almost surely) such that  $Y^{T_n}$  is a martingale, and set

$$\overline{X} := \phi \bullet X, \quad Q_n := Y_{T_n} / Y_0 \bullet P \sim P.$$

By applying assertion (b') to  $\overline{X}^{T_n}$  and  $Q_n \sim P$  (since  $\overline{X}^{T_n}$  is an  $\mathbb{F}$ -quasi-left-continuous element of  $\mathcal{M}(\mathbb{F}, Q_n)$ ), we conclude that  $(\phi \bullet (X - X^\tau))^{T_n} = \overline{X}^{T_n} - (\overline{X}^{T_n})^\tau$  satisfies  $\text{NUPBR}(\mathbb{G})$ . Hence, thanks -again- to Proposition A.2,  $\text{NUPBR}(\mathbb{G})$  for  $X - X^\tau$  follows immediately. This ends the proof of (b)  $\iff$  (c), and that of the proposition as well.  $\square$

**Theorem 2.21** *Suppose that  $\tau \in \mathcal{H}$  and  $\mathbb{F}$  is quasi-left-continuous. Then the following assertions are equivalent.*

- (a) *The thin set  $\{\tilde{Z} = 1 > Z_-\}$  is evanescent.*
- (b) *For every (bounded)  $\mathbb{F}$ -martingale  $X$ , the process  $X - X^\tau$  satisfies  $\text{NUPBR}(\mathbb{G})$ .*
- (b') *For any probability  $Q \sim P$  and every (bounded)  $\mathbb{F}$ -quasi-left-continuous  $X \in \mathcal{M}(Q, \mathbb{F})$ , the process  $X - X^\tau$  satisfies  $\text{NUPBR}(\mathbb{G})$ .*
- (c) *For every (bounded)  $X$  satisfying  $\text{NUPBR}(\mathbb{F})$ ,  $X - X^\tau$  satisfies the  $\text{NUPBR}(\mathbb{G})$ .*

*Proof* The proofs of both equivalences (b')  $\iff$  (c) and (b)  $\iff$  (b') follow the same arguments as the corresponding proofs in Theorem 2.20 (see parts 2) and 3)). Hence, we omit these proofs and the proof of (a)  $\implies$  (b) as well, as this latter one follows immediately from Theorem 2.20-(a) or Corollary 2.16-(d). Thus, the remaining part of the proof focuses on proving (a)  $\implies$  (b). To this end, we assume that assertion (b) holds, and recall that –when  $\mathbb{F}$  is a quasi-left-continuous filtration– any accessible  $\mathbb{F}$ -stopping time is predictable (see [10] or [15, Th. 4.26]). Then, since  $\mathbb{F}$  is a quasi-left-continuous filtration, any  $\mathbb{F}$ -martingale is quasi-left-continuous, and from Theorem 2.20 we deduce that the thin set,  $\{\tilde{Z} = 1 < Z_-\}$ , is predictable. Now take any  $\mathbb{F}$ -predictable stopping time  $T$  such that

$$[T] \subset \{\tilde{Z} = 1 > Z_-\}.$$

This implies that  $\{T < +\infty\} \subset \{\tilde{Z}_T = 1\}$ , and due to  $E(\tilde{Z}_T | \mathcal{F}_{T-}) = Z_{T-}$  on  $\{T < +\infty\}$ , we get

$$E(I_{\{T < +\infty\}}(1 - Z_{T-})) = E(I_{\{T < +\infty\}}(1 - \tilde{Z}_T)) = 0.$$

This leads to  $T = +\infty$   $P$ -a.s (since  $\{T < +\infty\} \subset \{Z_{T-} < 1\}$ ), and the proof of the theorem is completed.  $\square$

*Remark 2.22* The conclusion of Theorem 2.21 remains valid without the quasi-left-continuous assumption on the filtration  $\mathbb{F}$ . This general case, that can be found in the earlier version [1], requires more technical arguments.

*Remark 2.23* The proof of (b)  $\iff$  (c) in Theorem 2.14 is obvious (due to the fact that  $(1 - Z_-)^{-1}I_{\{Z_- < 1\}}$  is  $\mathbb{F}$ -locally bounded –see Proposition B.1–). Thus, the only parts that require proof are (a)  $\iff$  (b). The implication (b)  $\implies$  (a) can be formulated in a more abstract way, due to the simple fact that  $\mathcal{T}_a(S) - (\mathcal{T}_a(S))^\tau = S - S^\tau$  and  $\mathcal{T}_a(S)$  does not jump on  $\{\tilde{Z} = 1 > Z_-\}$ . Thus, in virtue of Proposition A.2, one can assume without loss of generality that  $S$  is an  $\mathbb{F}$ -quasi-left-continuous local martingale that does not jump on  $\{\tilde{Z} = 1 > Z_-\}$ , and prove that in this case  $S - S^\tau$  satisfies the  $\text{NUPBR}(\mathbb{G})$ . This is the aim of the next section in a more interesting and general manner, as it constructs explicitly a  $\mathbb{G}$ -deflator for any  $S - S^\tau$  as long as  $S \in \mathcal{M}_{loc}(\mathbb{F})$ , quasi-left-continuous, and orthogonal to  $V^\mathbb{F} - (V^\mathbb{F})^{p, \mathbb{F}}$  ( $V^\mathbb{F}$  is defined in (2.11)).

### 3 Explicit Deflators for a Class of $\mathbb{F}$ -Local Martingales

This section proposes an explicit construction of  $\mathbb{G}$ -deflators for  $M - M^\tau$ , when  $M$  spans a class of  $\mathbb{F}$ -quasi-left-continuous local martingales. The key mathematical idea behind this achievement lies in the exact relationship between the  $\mathbb{G}$ -compensator and the  $\mathbb{F}$ -compensator of a process with finite variation when both exists. This is the aim of the first subsection, while the second subsection states the results about deflators.

#### 3.1 Dual Predictable Projections under $\mathbb{G}$ and $\mathbb{F}$

In the following, we start our study by writing the  $\mathbb{G}$ -compensators/projections in terms of  $\mathbb{F}$ -compensators/projections respectively. Even though, the proofs of the results of this subsection are easy and not technical at all, we opted for delegating them to the Appendix for the reader's convenience.

**Lemma 3.1** *Suppose that  $\tau \in \mathcal{H}$ . Then, the following assertions hold.*

(a) *For any  $\mathbb{F}$ -adapted process  $V$ , with locally integrable variation we have*

$$I_{\llbracket \tau, +\infty \llbracket} \cdot V^{p, \mathbb{G}} = I_{\llbracket \tau, +\infty \llbracket} (1 - Z_-)^{-1} \cdot \left( (1 - \tilde{Z}) \cdot V \right)^{p, \mathbb{F}}, \quad (3.1)$$

and on  $\llbracket \tau, +\infty \llbracket$

$${}^{p, \mathbb{G}}(\Delta V) = (1 - Z_-)^{-1} {}^{p, \mathbb{F}}\left((1 - \tilde{Z})\Delta V\right). \quad (3.2)$$

(b) *For any  $\mathbb{F}$ -local martingale  $M$ , one has, on  $\llbracket \tau, +\infty \llbracket$*

$${}^{p, \mathbb{G}}\left(\frac{\Delta M}{1 - \tilde{Z}}\right) = \frac{{}^{p, \mathbb{F}}(\Delta M I_{\{\tilde{Z} < 1\}})}{1 - Z_-}, \quad \text{and} \quad {}^{p, \mathbb{G}}\left(\frac{1}{1 - \tilde{Z}}\right) = \frac{{}^{p, \mathbb{F}}(I_{\{\tilde{Z} < 1\}})}{1 - Z_-}. \quad (3.3)$$

(c) *For any quasi-left-continuous  $\mathbb{F}$ -local martingale  $M$ , one has*

$${}^{p, \mathbb{G}}\left((\Delta M)(1 - \tilde{Z})^{-1} I_{\llbracket \tau, +\infty \llbracket}\right) = 0. \quad (3.4)$$

The next lemma focuses on the integrability of the process  $(1 - \tilde{Z})^{-1} I_{\llbracket \tau, +\infty \llbracket}$  with respect to any process with  $\mathbb{F}$ -locally integrable variation. As a result, we complete our comparison of  $\mathbb{G}$  and  $\mathbb{F}$  compensators. Recall that, due to [12, Chapter XX],  $\tilde{Z} = Z$  on  $\llbracket \tau, +\infty \llbracket$ .

**Lemma 3.2** *Let  $\tau$  be an honest time and  $V$  be a càdlàg and  $\mathbb{F}$ -adapted process with finite variation. Then, the following assertions hold.*

(a) *The process*

$$U := (1 - Z)^{-1} I_{\llbracket \tau, +\infty \llbracket} \cdot V, \quad (3.5)$$

is a well defined process, that is  $\mathbb{G}$ -adapted, càdlàg and has finite variation.

(b) If  $V$  belongs to  $\mathcal{A}_{loc}(\mathbb{F})$  (respectively to  $\mathcal{A}(\mathbb{F})$ ), then  $U \in \mathcal{A}_{loc}(\mathbb{G})$  (respectively  $U \in \mathcal{A}(\mathbb{G})$ ) and

$$U^{p,\mathbb{G}} = I_{\tau,+\infty}[(1 - Z_-)^{-1} \cdot (I_{\{\tilde{Z} < 1\}} \cdot V)^{p,\mathbb{F}}]. \quad (3.6)$$

(c) Suppose furthermore that  $\tau$  is finite almost surely. Then,  $I_{\tau,+\infty} \cdot V \in \mathcal{A}_{loc}(\mathbb{G})$  if and only if  $(1 - \tilde{Z}) \cdot V \in \mathcal{A}_{loc}(\mathbb{F})$ .

(d) Suppose furthermore that  $\tau$  is finite almost surely, and  $V$  is a nondecreasing and  $\mathbb{F}$ -predictable process. Then, for any  $\mathbb{F}$ -predictable process  $\varphi \geq 0$ ,  $\varphi I_{\tau,+\infty} \cdot V \in \mathcal{A}_{loc}^+(\mathbb{G})$  iff  $(1 - Z_-)\varphi \cdot V \in \mathcal{A}_{loc}^+(\mathbb{F})$  iff  $\varphi I_{\{Z_- < 1\}} \cdot V \in \mathcal{A}_{loc}^+(\mathbb{F})$ .

### 3.2 Construction of Deflators

Herein, we start by introducing a deflator-candidate as follows.

**Proposition 3.3** Suppose that  $\tau \in \mathcal{H}$  and consider the  $\mathbb{G}$ -local martingale

$$\hat{m} := I_{\tau,+\infty} \cdot m + (1 - Z_-)^{-1} I_{\tau,+\infty} \cdot \langle m \rangle^{\mathbb{F}}, \quad (3.7)$$

and the process

$$W^{\mathbb{G}} := \left( (1 - Z_-)(1 - \tilde{Z}) \right)^{-1} I_{\tau,+\infty} \cdot [m, m]. \quad (3.8)$$

Then, the following assertions hold.

- 1) The nondecreasing and  $\mathbb{G}$ -optional process  $W^{\mathbb{G}}$  belongs to  $\mathcal{A}_{loc}^+(\mathbb{G})$ .
- 2) The  $\mathbb{G}$ -local martingale

$$L^{\mathbb{G}} := (1 - Z_-)^{-1} I_{\tau,+\infty} \cdot \hat{m} + W^{\mathbb{G}} - (W^{\mathbb{G}})^{p,\mathbb{G}}, \quad (3.9)$$

satisfies the following properties:

- (2-a)  $\mathcal{E}(L^{\mathbb{G}}) > 0$  (or equivalently  $1 + \Delta L^{\mathbb{G}} > 0$ ) and  $I_{[0,\tau]} \cdot L^{\mathbb{G}} = 0$ .
- (2-b) For any  $M \in \mathcal{M}_{0,loc}(\mathbb{F})$ , we have

$$[L^{\mathbb{G}}, \widehat{M}] \in \mathcal{A}_{loc}(\mathbb{G}) \quad \left( \text{i.e. } \langle L^{\mathbb{G}}, \widehat{M} \rangle^{\mathbb{G}} \text{ exists} \right), \quad (3.10)$$

where  $\widehat{M}$  is defined in (2.5).

*Proof* Thanks to Lemma 2.6-(b),  $(1 - Z_-)^{-1} I_{\tau,+\infty}$  is  $\mathbb{G}$ -locally bounded. Thus, by combining this fact with  $[m, m] \in \mathcal{A}_{loc}^+(\mathbb{F})$  and Lemma 3.2-(b), we conclude that  $W^{\mathbb{G}} = (1 - Z_-)^{-1} (1 - \tilde{Z})^{-1} I_{\tau,+\infty} \cdot [m, m] \in \mathcal{A}_{loc}^+(\mathbb{G})$ , and subsequently assertion (1) holds. Thus, the process  $L^{\mathbb{G}}$ —given in (3.9)—is a well defined  $\mathbb{G}$ -local martingale. The rest of this proof focuses on proving the properties (2-a) and (2-b). To this end, by combining Lemma 3.1-(b), the fact

that  $\Delta(V^{p,\mathbb{H}}) = {}^{p,\mathbb{H}}(\Delta V)$  for any process  $V$  with locally integrable variation and any filtration  $\mathbb{H}$ , and  $\Delta m = \tilde{Z} - Z_-$ , on  $]\tau, +\infty[$  we calculate

$$\begin{aligned} \Delta L^{\mathbb{G}} &= (1 - Z_-)^{-1} \Delta \hat{m} + \Delta W^{\mathbb{G}} - \Delta(W^{\mathbb{G}})^{p,\mathbb{G}} \\ &= \frac{\Delta m}{1 - Z_-} + \frac{\Delta \langle m \rangle^{\mathbb{F}}}{(1 - Z_-)^2} + \frac{(\Delta m)^2}{(1 - \tilde{Z})(1 - Z_-)} - {}^{p,\mathbb{G}} \left( \frac{(\Delta m)^2}{(1 - \tilde{Z})(1 - Z_-)} \right) \\ &= \frac{\Delta m}{1 - \tilde{Z}} + \frac{\Delta \langle m \rangle^{\mathbb{F}}}{(1 - Z_-)^2} - \frac{{}^{p,\mathbb{F}} \left( (\Delta m)^2 I_{\{\tilde{Z} < 1\}} \right)}{(1 - Z_-)^2} = -1 + \frac{1 - Z_-}{1 - \tilde{Z}} + {}^{p,\mathbb{F}} \left( I_{\{\tilde{Z} = 1\}} \right) \end{aligned}$$

Therefore,  $1 + \Delta L^{\mathbb{G}} = I_{]\tau, +\infty[} \left[ \frac{1 - Z_-}{1 - \tilde{Z}} + {}^{p,\mathbb{F}} \left( I_{\{\tilde{Z} = 1\}} \right) \right] + I_{[0, \tau]} > 0$ . This proves the property (2-a). In order to prove the property (2-b), we consider a quasi-left-continuous  $\mathbb{F}$ -local martingale  $M$ . Then, it is obvious that this quasi-left-continuous assumption implies that  $\langle m, M \rangle^{\mathbb{F}}$  is continuous and  $[X, M] \equiv 0$  for any  $\mathbb{G}$ -predictable process with finite variation  $X$ . As a result, we derive

$$\begin{aligned} [L^{\mathbb{G}}, \widehat{M}] &= [L^{\mathbb{G}}, M - M^{\tau}] = (1 - Z_-)^{-1} I_{]\tau, +\infty[} \cdot [m, M] + [W^{\mathbb{G}}, M] \\ &= \frac{1}{1 - Z_-} I_{]\tau, +\infty[} \cdot [m, M] + \frac{\Delta m}{(1 - Z_-)(1 - \tilde{Z})} I_{]\tau, +\infty[} \cdot [m, M] \\ &= (1 - \tilde{Z})^{-1} I_{]\tau, +\infty[} \cdot [m, M]. \end{aligned} \quad (3.11)$$

Therefore, since  $[m, M] \in \mathcal{A}_{loc}(\mathbb{F})$ , the property (2-b) follows immediately from combining the above equality and Lemma 3.2-(b). This ends the proof of the proposition.  $\square$

Below, we elaborate our main results deflator for “the part-after- $\tau$ ”.

**Theorem 3.4** *Let  $\tau \in \mathcal{H}$  be a finite almost surely and  $L^{\mathbb{G}}$  be defined by (3.9). Then, the following assertions hold.*

- (a) *If  $M$  is a quasi-left-continuous  $\mathbb{F}$ -local martingale such that  $\sum \Delta M I_{\{\tilde{Z}=1 > Z_-\}}$  is also an  $\mathbb{F}$ -local martingale, then  $\mathcal{E}(L^{\mathbb{G}})(M - M^{\tau})$  is a  $\mathbb{G}$ -local martingale.*
- (b) *For any quasi-left-continuous  $\mathbb{F}$ -local martingale,  $M$ , such that  $\{\tilde{Z} = 1 > Z_-\} \cap \{\Delta M \neq 0\}$  is evanescent,  $\mathcal{E}(L^{\mathbb{G}})(M - M^{\tau})$  is a  $\mathbb{G}$ -local martingale.*

*Proof* Let  $M$  be a quasi-left-continuous  $\mathbb{F}$ -local martingale such that  $V := \sum \Delta M I_{\{\tilde{Z}=1 > Z_-\}}$  is an  $\mathbb{F}$ -local martingale. As a result, we get

$$0 = \left( (1 - Z_-) \cdot V \right)^{p,\mathbb{F}} = \left( I_{\{\tilde{Z}=1\}} \cdot [m, M] \right)^{p,\mathbb{F}}.$$

Therefore, by combining this equation, (3.11) and Lemma 3.2-(b), we obtain

$$\begin{aligned} M - M^{\tau} + \langle L^{\mathbb{G}}, M - M^{\tau} \rangle^{\mathbb{G}} &= M - M^{\tau} + \left( \frac{I_{]\tau, +\infty[}}{1 - \tilde{Z}} \cdot [m, M] \right)^{p,\mathbb{G}} \\ &= M - M^{\tau} + \frac{I_{]\tau, +\infty[}}{1 - Z_-} \cdot \langle m, M \rangle^{\mathbb{F}} - \frac{I_{]\tau, +\infty[}}{1 - Z_-} \cdot \left( I_{\{\tilde{Z}=1\}} \cdot [m, M] \right)^{p,\mathbb{F}} \\ &= M - M^{\tau} + (1 - Z_-)^{-1} I_{]\tau, +\infty[} \cdot \langle m, M \rangle^{\mathbb{F}} = \widehat{M} \in \mathcal{M}_{loc}(\mathbb{G}). \end{aligned}$$



Thus, assertion (a) follows immediately from this combined with Ito's formula applied to  $(M - M^\tau)\mathcal{E}(L^\mathbb{G})$ . Assertion (b) follows obviously from assertion (a), and the proof of the theorem is completed.  $\square$

As a consequence of this theorem, we describe a class of  $\mathbb{F}$ -quasi-left-continuous processes for which the NUPBR property is preserved for the “part-after- $\tau$ ”.

**Corollary 3.5** *Suppose that  $\tau \in \mathcal{H}$  is finite almost surely, and  $S$  is  $\mathbb{F}$ -quasi-left-continuous. If  $(S, \sum \Delta S I_{\{\Delta S \neq 0\}})$  satisfies the NUPBR( $\mathbb{F}$ ), then  $S - S^\tau$  satisfies the NUPBR( $\mathbb{G}$ ).*

*Proof* The proof follows immediately from a combination of Theorem 3.4(a), Proposition A.2 (see the appendix), and the fact that

$$\{\tilde{Z} = 1 > Z_-\} = \{\tilde{Z}^Q = 1 > Z_-^Q\} \quad \text{for any } Q \sim P, \quad (3.12)$$

where  $\tilde{Z}_t^Q := Q(\tau \geq t | \mathcal{F}_t)$  and  $Z_t^Q := Q(\tau > t | \mathcal{F}_t)$ . This last fact is an immediate application of Theorem 86 of [11] by taking on the one hand  $X = I_{\{\tilde{Z}=0\}}$  and  $Y = I_{\{\tilde{Z}^Q=0\}}$ , and on the other hand  $X = I_{\{Z_-=0\}}$  and  $Y = I_{\{Z_-^Q=0\}}$ .  $\square$

#### 4 Proof of Theorems 2.14 and 2.18

This section focuses on the proofs of Theorems 2.14 and 2.18. Both proofs are based essentially on the predictable characteristics of  $S$  under  $\mathbb{F}$  and  $\mathbb{G}$ . This section is divided into three subsections. The first subsection recalls the predictable characteristics, and proposes afterwards a functional  $\psi$ , which is intimately related to the set  $\{\tilde{Z} = 1 > Z_-\}$ . This  $\psi$  quantifies the part responsible for  $\mathbb{G}$ -arbitrages. The second and third subsections are devoted to the proof of Theorems 2.14 and 2.18 respectively.

##### 4.1 Predictable Characteristics of $S$ and the Functional $\psi$

To the process  $S$ , we associate its random measure of jumps  $\mu(dt, dx) := \sum_{u>0} I_{\{\Delta S_u \neq 0\}} \delta_{(u, \Delta S_u)}(dt, dx)$ . For any nonnegative product-measurable functional  $H(t, \omega, x)$ , we define the process  $H \star \mu$  and a  $\sigma$ -finite measure  $M_\mu^P$  on the measurable space  $(\Omega \times \mathbb{R}^+ \times \mathbb{R}^d, \mathcal{F}_\infty \otimes \mathcal{B}(\mathbb{R}^+) \otimes \mathcal{B}(\mathbb{R}^d))$  by

$$H \star \mu_t := \int_0^t \int_{\mathbb{R}^d} H(u, x) \mu(du, dx) \quad \text{and} \quad M_\mu^P(H) := E[H \star \mu_\infty]. \quad (4.1)$$

Throughout the rest of the paper, for any filtration  $\mathbb{H}$ , we denote

$$\tilde{\mathcal{O}}(\mathbb{H}) := \mathcal{O}(\mathbb{H}) \otimes \mathcal{B}(\mathbb{R}^d), \quad \tilde{\mathcal{P}}(\mathbb{H}) := \mathcal{P}(\mathbb{H}) \otimes \mathcal{B}(\mathbb{R}^d),$$

and  $M_\mu^P(W | \tilde{\mathcal{P}}(\mathbb{H}))$ , for a nonnegative or bounded functional  $W$ , is the unique  $\tilde{\mathcal{P}}(\mathbb{H})$ -measurable functional  $Y$  satisfying  $M_\mu^P(YU) = M_\mu^P(WU)$  for any bounded

and  $\tilde{\mathcal{P}}(\mathbb{H})$ -measurable functional  $U$ . The random measure  $\nu(dt, dx)$  is the unique  $\mathbb{F}$ -predictable random measure satisfying  $(H \star \mu)^{p, \mathbb{F}} = H \star \nu$ , for any  $\tilde{\mathcal{P}}(\mathbb{F})$ -measurable and nonnegative  $H$ . There is a version of  $\nu$  taking the form of  $\nu(dt, dx) = F_t(dx) dA_t$  where  $A$  is a nondecreasing and  $\mathbb{F}$ -predictable process and  $F(dx)$  is an  $\mathbb{F}$ -predictable kernel. Then, the canonical decomposition of  $S$  is

$$S = S_0 + S^c + h \star (\mu - \nu) + b \bullet A + (x - h) \star \mu, \quad (4.2)$$

where  $h(x) := xI_{\{|x| \leq 1\}}$ ,  $S^c$  is the continuous  $\mathbb{F}$ -local martingale part of  $S$ ,  $b$  is an  $\mathbb{F}$ -predictable process,  $h \star (\mu - \nu)$  is the unique pure jumps  $\mathbb{F}$ -local martingale with jumps given by  $h(\Delta S)I_{\{\Delta S \neq 0\}}$ , and there exists an  $\mathbb{F}$ -predictable matrix process,  $c$ , such that  $\langle S^c, S^c \rangle^{\mathbb{F}} = c \bullet A$ . Then,

the quadruplet  $(b, c, F, A)$  is the  $\mathbb{F}$  – predictable characteristics of  $S$ . (4.3)

These characteristics parameterize the model  $(S, \mathbb{F}, P)$ , and will be used frequently throughout the remaining part of the paper.

The following identifies explicitly the source of  $\mathbb{G}$ -arbitrage for  $S - S^\tau$  denoted by the functional  $\psi$ , and gives some of its properties.

**Lemma 4.1** *Consider*

$$f_m := M_\mu^P(\Delta m | \tilde{\mathcal{P}}(\mathbb{F})), \quad \text{and} \quad \psi := M_\mu^P(I_{\{\tilde{Z} < 1\}} | \mathcal{P}(\mathbb{F})). \quad (4.4)$$

*Then, the following hold.*

(a) *The process  $(f_m)^2 \star \mu$  belongs to  $\mathcal{A}_{loc}^+(\mathbb{F})$ , and there exist  $\beta_m \in L(S^c)$  and  $m^\perp \in \mathcal{M}_{loc}(\mathbb{F})$  (i.e.  $(\beta_m)^{tr} c \beta_m \bullet A \in \mathcal{A}_{loc}^+(\mathbb{F})$ ) such that  $[S^c, m^\perp] \equiv 0$  and*

$$m = m_0 + \beta_m \bullet S^c + m^\perp. \quad (4.5)$$

(b) *We have  $\{\psi = 0\} = \{Z_- + f_m = 1\} \subset \{\tilde{Z} = 1\}$ ,  $M_\mu^P$  – a.e. or equivalently*

$$\{\psi = 0\} = \{Z_- + f_m = 1\} \subset \{\tilde{Z} = 1\} \quad \text{on } \{\Delta S \neq 0\}. \quad (4.6)$$

(c) *The nondecreasing process  $I_{\{\psi=0\} \& Z_- < 1\}} \star \mu$  is càdlàg and  $\mathbb{F}$ -locally integrable under any probability measure  $Q$ .*

*Proof* Assertion (a) is proved in [1]. Thus, we address assertions (b) and (c).

1) Here, we prove assertion (b). Recall that we always have  $E[W \star \mu_\infty] = E[M_\mu^P(W | \tilde{\mathcal{P}}(\mathbb{F})) \star \nu_\infty]$ , for any non-negative  $\tilde{\mathcal{O}}(\mathbb{F})$ -measurable functional  $W$ .

Thus, since  $M_\mu^P(\tilde{Z} | \tilde{\mathcal{P}}(\mathbb{F})) = Z_- + f_m$  and  $1 - \tilde{Z} \leq I_{\{\tilde{Z} < 1\}}$ , we derive

$$\begin{aligned} 0 \leq 1 - Z_- - f_m &\leq \psi \quad M_\mu^P - \text{a.e.} \quad \text{and} \\ E[(1 - \tilde{Z})I_{\{Z_- + f_m = 1\}} \star \mu_\infty] &= E[(1 - Z_- - f_m)I_{\{Z_- + f_m = 1\}} \star \mu_\infty] = 0. \end{aligned}$$

These clearly prove that, on one hand, we have

$$\{\psi = 0\} \subset \{Z_- + f_m = 1\} \subset \{\tilde{Z} = 1\}, \quad M_\mu^P - \text{a.e.}$$

On the other hand, we derive

$$E \left[ I_{\{Z_- + f_m = 1\}} \psi \star \mu_\infty \right] = E \left[ I_{\{Z_- + f_m = 1\}} I_{\{\tilde{Z} < 1\}} \star \mu_\infty \right] = 0.$$

This proves that  $\{Z_- + f_m = 1\} \subset \{\psi = 0\}$ ,  $M_\mu^P - a.e.$ , and the proof of the assertion (a) is completed.

2) The proof of the assertion b) follows immediately from Lemma 2.13 (where  $V^\mathbb{F}$  is defined and we recall here  $V^\mathbb{F} := \sum I_{\{\tilde{Z}=1 > Z_-\}}$  for the reader's convenience) and the following inequality (which is due to (4.6))

$$HI_{\{\psi=0 \ \& \ Z_- < 1\}} \star \mu \leq HI_{\{\tilde{Z}=1 > Z_-\}} \star \mu \leq \sum HI_{\{\tilde{Z}=1 > Z_-\}} =: H \cdot V^\mathbb{F},$$

for  $H$  nonnegative and bounded. This ends the proof of the lemma.  $\square$

We end this subsection with providing the  $\mathbb{G}$ -predictable characteristics of  $S - S^\tau$  as follows. Throughout the rest of the paper, we put  $\mu^\mathbb{G}(dt, dx) := I_{\{t > \tau\}} \mu(dt, dx)$ , and deduce that  $\nu^\mathbb{G}$ —its  $\mathbb{G}$ -compensator—is given by

$$\nu^\mathbb{G}(dt, dx) := I_{\{t > \tau\}} \left[ 1 - f_m(x, t)(1 - Z_{t-})^{-1} \right] \nu(dt, dx). \quad (4.7)$$

Furthermore, the  $\mathbb{G}$ -canonical decomposition of  $S - S^\tau$  is given by

$$\begin{aligned} S - S^\tau &= \widehat{S}^c + h \star (\mu^\mathbb{G} - \nu^\mathbb{G}) + bI_{\llbracket \tau, +\infty \rrbracket} \cdot A - \frac{c\beta_m}{1 - Z_-} I_{\llbracket \tau, +\infty \rrbracket} \cdot A \\ &\quad - h \frac{f_m}{1 - Z_-} I_{\llbracket \tau, +\infty \rrbracket} \star \nu + (x - h) \star \mu^\mathbb{G}, \end{aligned}$$

where  $\widehat{S}^c$  is defined by (2.5). This decomposition clearly states that the  $\mathbb{G}$ -predictable characteristics of  $S - S^\tau$ ,  $(b^\mathbb{G}, c^\mathbb{G}, F^\mathbb{G}, A^\mathbb{G})$ , are given by

$$\begin{aligned} b^\mathbb{G} &:= b - \left[ \int h(x) f_m(x) F(x) + c\beta_m \right] (1 - Z_-)^{-1} I_{\{Z_- < 1\}}, \quad c^\mathbb{G} := c \\ F^\mathbb{G}(dx) &:= \left( 1 - \frac{f_m(x, t)}{1 - Z_-} \right) I_{\{Z_- < 1\}} F(dx), \quad A^\mathbb{G} := A - A^\tau. \end{aligned} \quad (4.8)$$

#### 4.2 Proof of Theorem 2.14

This section proves this theorem. To this end, we start by singling out, in the following remark, the simplest parts of the theorem, and the key ideas for the proof of the difficult part(s) as well.

*Remark 4.2* 1) Since  $(1 - Z_-)^l I_{\{Z_- < 1\}}$  (for any  $l \in \mathbb{R}$ ) is  $\mathbb{F}$ -locally bounded (see Lemma 2.6), it is easy to see that  $I_{\{Z_- < 1\}} \cdot X$  satisfies the NUPBR( $\mathbb{F}$ ) if and only if  $(1 - Z_-) \cdot X$  does, for any  $\mathbb{F}$ -semimartingale  $X$ . As a result, on one hand, the proof of (b)  $\iff$  (c) follows immediately from this fact. On the other hand, as it is mentioned in Remark 2.23, it is very clear that

$$S - S^\tau = \mathcal{T}_a(S) - (\mathcal{T}_a(S))^\tau \quad \text{and} \quad \{\Delta \mathcal{T}_a(S) \neq 0\} \cap \{\tilde{Z} = 1 > Z_-\} = \emptyset,$$

and the proof of (b)  $\implies$  (a) follows from combining these with Corollary 3.5. Therefore, the rest of this section prepares and delivers afterwards the proof of (a)  $\implies$  (b), which is the most technical and difficult part of the theorem. 2) The proof of (a)  $\implies$  (b) relies on applying Theorem A.1 adequately. Hence, the first task in proving this part resides in we guessing/getting the pair  $(\beta^{(1)}, f^{(1)})$  for  $(\mathcal{T}_a(S), \mathbb{F})$  from  $(\beta^{\mathbb{G}}, f^{\mathbb{G}})$  associated to  $(S - S^\tau, \mathbb{G})$ . Thanks to Proposition A.3, the next lemma prepares the ground for this goal by providing equivalent statement to the NUPBR of  $(S - S^\tau, \mathbb{G})$ , using the  $\mathbb{F}$ -predictable functionals only. After this step, we will prove that the chosen pair fulfills the conditions (A.1)-(A.2)-(A.3) that correspond to the model  $(I_{\{Z_- < 1\}} \bullet \mathcal{T}_a(S), \mathbb{F})$ .

**Lemma 4.3** *Let  $\Phi_\alpha(f)$  (for  $\alpha > 0$ ) be defined by*

$$\Phi_\alpha(f) := (f - 1)^2 I_{\{|f-1| \leq \alpha\}} + |f - 1| I_{\{|f-1| > \alpha\}}, \quad \text{for any } f \in \tilde{\mathcal{P}}(\mathbb{F}). \quad (4.9)$$

*Then,  $(S - S^\tau)$  satisfies the NUPBR( $\mathbb{G}$ ) if and only if there exists a pair,  $(\beta^{\mathbb{F}}, f^{\mathbb{F}})$ , of  $\mathbb{F}$ -predictable process and  $\tilde{\mathcal{P}}(\mathbb{F})$ -predictable functional, such that  $f^{\mathbb{F}} > 0$   $M_\mu^P - a.e.$ ,*

$$(\beta^{\mathbb{F}})^{tr} c\beta^{\mathbb{F}} I_{\{Z_- < 1\}} \bullet A + \Phi_\alpha(f^{\mathbb{F}})(1 - Z_- - f_m) I_{\{Z_- < 1\}} \star \mu \in \mathcal{A}_{loc}^+(\mathbb{F}), \quad (4.10)$$

*and  $P \otimes A - a.e.$  on  $\{Z_- < 1\}$ , we have*

$$\int |x f^{\mathbb{F}}(x) \left(1 - \frac{f_m(x)}{1 - Z_-}\right) - h(x)| F(dx) < +\infty \quad \text{and} \quad (4.11)$$

$$b + c \left( \beta^{\mathbb{F}} - \frac{\beta_m}{1 - Z_-} \right) + \int \left[ x f^{\mathbb{F}}(x) \left(1 - \frac{f_m(x)}{1 - Z_-}\right) - h(x) \right] F(dx) \equiv 0. \quad (4.12)$$

*Proof* In virtue of Theorem A.1, by using the  $\mathbb{G}$ -predictable characteristics of  $S - S^\tau$ , given in (4.8), we deduce that  $(S - S^\tau)$  satisfies the NUPBR( $\mathbb{G}$ ) iff there exists a pair of  $\mathbb{G}$ -predictable functionals  $(\beta^{\mathbb{G}}, f^{\mathbb{G}})$  such that  $f^{\mathbb{G}} > 0$ ,

$$(\beta^{\mathbb{G}})^{tr} c\beta^{\mathbb{G}} I_{\llbracket \tau, +\infty \rrbracket} \bullet A + \sqrt{(f^{\mathbb{G}} - 1)^2 I_{\llbracket \tau, +\infty \rrbracket}} \star \mu \in \mathcal{A}_{loc}^+(\mathbb{G}), \quad (4.13)$$

and  $P \otimes A - a.e.$  on  $\llbracket \tau, +\infty \rrbracket$ ,

$$\int |x f^{\mathbb{G}}(x) - h(x)| (1 - Z_- - f_m) F(dx) < +\infty, \quad \text{and} \quad (4.14)$$

$$0 \equiv b + c \left( \beta^{\mathbb{G}} - \frac{\beta_m}{1 - Z_-} \right) + \int \left[ x f^{\mathbb{G}}(x) \left(1 - \frac{f_m(x)}{1 - Z_-}\right) - h(x) \right] F(dx). \quad (4.15)$$

Furthermore, to this pair  $(\beta^{\mathbb{G}}, f^{\mathbb{G}})$ , Proposition A.3 guarantees the existence of a pair of  $\mathbb{F}$ -predictable functionals  $(\beta^{\mathbb{F}}, f^{\mathbb{F}})$  such that  $f^{\mathbb{F}} > 0$  and

$$(\beta^{\mathbb{G}}, f^{\mathbb{G}}) = (\beta^{\mathbb{F}}, f^{\mathbb{F}}) \quad \text{on } \llbracket \tau, +\infty \rrbracket. \quad (4.16)$$

Therefore, by inserting (4.16) in the three conditions, (4.13), (4.14) and (4.15), and using afterwards Lemma 3.2-(d) and Proposition B.3 (precisely assertions (b), (c) and (d)), the proof of the lemma follows immediately.  $\square$

*Remark 4.4* To guess the pair  $(\beta^{(0)}, f^{(0)})$  for the model  $(S^{(0)} := I_{\{Z_- < 1\}} \cdot \mathcal{T}_a(S), \mathbb{F})$  from the pair,  $(\beta^{\mathbb{F}}, f^{\mathbb{F}})$ , provided by the above lemma, we need to derive the  $\mathbb{F}$ -predictable characteristics of the model. Thus, we start by getting the random measure associated to the jumps of  $S^{(0)}$  as  $\mu^{(0)}(dt, dx) := I_{\{\tilde{Z} < 1 \text{ \& } Z_- < 1\}} \mu(dt, dx)$ , and its  $\mathbb{F}$ -compensator  $\nu^{(0)}$  given by

$$\nu^{(0)}(dt, dx) := \psi(t, x) I_{\{Z_{t-} < 1\}} \nu(dt, dx). \quad (4.17)$$

Then, by combining this with (4.2),  $h I_{\{\tilde{Z}=1 > Z_-\}} \star \mu \in \mathcal{A}_{loc}(\mathbb{F})$ , and  $\left(h I_{\{\tilde{Z}=1 > Z_-\}} \star \mu\right)^{p, \mathbb{F}} = h \psi I_{\{Z_- < 1\}} \star \nu$  to derive the canonical decomposition of  $(S^{(0)}, \mathbb{F})$  and get its  $\mathbb{F}$ -predictable characteristics,  $(b^{(0)}, c^{(0)}, F^{(0)}(dx), A^{(0)})$ , as follows

$$\begin{aligned} b^{(0)} &:= b - \int h(x) \psi(x) F(dx), & c^{(0)} &:= c \\ F^{(0)}(dx) &:= \psi(x) F(dx), & A^{(0)} &:= I_{\{Z_- < 1\}} \cdot A. \end{aligned} \quad (4.18)$$

Then, in virtue of Theorem A.1,  $(S^{(0)}, \mathbb{F})$  satisfies the NUPBR if and only if there exists a pair  $(\beta^{(0)}, f^{(0)})$  satisfying  $f^{(0)} > 0$ , (A.1) and (A.2) hold, and after simplifications using (4.18),

$$b + c\beta^{(0)} + \int \left[ x f^{(0)}(x) \psi(x) - h(x) \right] F(dx) \equiv 0. \quad (4.19)$$

Therefore, by comparing this equation to (4.12), one can easily conclude that the only pair,  $(\beta^{(0)}, f^{(0)})$ , that comes from the pair  $(\beta^{\mathbb{F}}, f^{\mathbb{F}})$  is given by

$$\beta^{(0)} := \left( \beta^{\mathbb{F}} - \frac{\beta_m}{1 - Z_-} \right) I_{\{Z_- < 1\}}, \text{ \& } f^{(0)} := f^{\mathbb{G}}(x) \left( 1 - \frac{f_m(x)}{1 - Z_-} \right) \psi^{-1} I_{\{\psi > 0 \text{ \& } Z_- < 1\}}.$$

This (apparently) unique choice leads to a major obstacle, as we have no information regarding the integrability of  $\psi^{-1} I_{\{\psi > 0\}}$ . However, this also explains that finding an “equivalent” model that will allow us to control this integrability problem imposes itself. This is the aim of the following.

**Proposition 4.5** *Let  $\mathcal{T}_a(S)$  be defined in (2.12) and consider*

$$m^{(1)} := I_{\{\psi=0 \text{ \& } Z_- < 1\}} \star (\mu - \nu) \quad \text{and} \quad S^{(1)} := I_{\{Z_- < 1\}} \cdot S - [S, m^{(1)}]. \quad (4.20)$$

*Then,  $S^{(0)} := I_{\{Z_- < 1\}} \cdot \mathcal{T}_a(S)$  satisfies the NUPBR( $\mathbb{F}$ ) if and only if  $S^{(1)}$  satisfies the NUPBR( $\mathbb{F}$ ).*

The proof of this proposition is delegated to the appendix for the reader's convenience. Now, we are in the stage of proving Theorem 2.14.

*Proof of Theorem 2.14* Suppose that  $S - S^\tau$  satisfies NUPBR( $\mathbb{G}$ ). Then, due to Lemma 4.3, we deduce the existence of  $(\beta^{\mathbb{F}}, f^{\mathbb{F}})$  satisfying  $f^{\mathbb{F}} > 0$ , (4.10), (4.11) and (4.12). Thanks to Proposition 4.5, this proof will be completed as

soon as we prove that  $(S^{(1)}, \mathbb{F})$  satisfies the NUPBR. This is the aim of the rest of the proof. To this end, we put  $\Sigma_1 := \{Z_- < 1 \text{ \& } \psi > 0\}$ ,  $\tilde{\Omega} := \Omega \times [0, +\infty)$ ,

$$\beta := (\beta^{\mathbb{F}} - \frac{\beta_m}{1 - Z_-})I_{\{Z_- < 1\}}, \quad f := f^{\mathbb{F}}(x) \left(1 - \frac{f_m(x)}{1 - Z_-}\right) I_{\Sigma_1} + I_{\tilde{\Omega} \setminus \Sigma_1}. \quad (4.21)$$

It is obvious that  $f > 0$ . To apply Theorem A.1 using the above pair  $(\beta, f)$ , we need to derive the predictable characteristics  $(S^{(1)}, \mathbb{F})$ . Thus, we start by getting the random measure for the jumps of this model by  $\mu^{(1)}(dt, dx) := \mu_{S^{(1)}}(dt, dx) = I_{\{\psi(t, x) > 0\}} I_{\{Z_{t-} < 1\}} \mu(dx, dt)$ , and its  $\mathbb{F}$ -compensator

$$\nu^{(1)}(dt, dx) := I_{\Sigma_1}(tx) \nu(dt, dx), \quad \Sigma_1 := \{\psi > 0 \text{ \& } Z_- < 1\}. \quad (4.22)$$

Then, again, combining this with (4.2), and  $(hI_{\{\psi=0 \text{ \& } Z_- < 1\}} \star \mu)^{p, \mathbb{F}} = hI_{\{\psi=0 \text{ \& } Z_- < 1\}} \star \nu$ , we derive easily the canonical decomposition of the model and get its predictable characteristics,  $(b^{(1)}, c^{(1)}, F^{(1)}(dx), A^{(1)})$ , as follows:

$$\begin{aligned} b^{(1)} &:= b - \int h(x) I_{\{\psi(x)=0\}} F(dx), \quad c^{(1)} := c, \\ F^{(1)}(dx) &:= I_{\{\psi(t, x) > 0\}} F(dx), \quad A^{(1)} := I_{\{Z_- < 1\}} \cdot A. \end{aligned} \quad (4.23)$$

It is obvious that, by plugging (4.21) and (4.23) into (4.11) and (4.12), we get

$$\begin{aligned} \int |xf^{(1)}(x) - h(x)| F^{(1)}(dx) &< +\infty \quad P \otimes A^{(0)} - a.e. \quad \text{and} \\ b^{(1)} + c\beta^{(0)} + \int [xf^{(1)}(x) - h(x)] F^{(1)}(dx) &\equiv 0, \quad P \otimes A^{(0)} - a.e.. \end{aligned}$$

Thus, we focus in the rest of this proof on proving the integrability condition (A.1) for the pair  $(\beta, f)$ . Due to the local boundedness of  $(1 - Z_-)^{-2} I_{\{Z_- < 1\}}$  (see Lemma 2.6) and  $(\beta_m)^{tr} c \beta_m \cdot A + (\beta^{\mathbb{F}})^{tr} c \beta^{\mathbb{F}} \cdot A \in \mathcal{A}_{loc}^+(\mathbb{F})$  (see Lemma 4.1 and (4.10)), we deduce that  $\beta^{tr} c \beta \cdot A \in \mathcal{A}_{loc}^+(\mathbb{F})$ . Therefore, now we deal with  $\sqrt{(f - 1)^2 \star \mu^{(1)}} \in \mathcal{A}_{loc}^+(\mathbb{F})$ . Then,

$$f - 1 = (f^{\mathbb{F}} - 1) \left(1 - \frac{f_m(x)}{1 - Z_-}\right) I_{\Sigma_1} - \frac{f_m(x)}{1 - Z_-} I_{\Sigma_1}, \quad \Sigma_1 := \{\psi > 0 \text{ \& } Z_- < 1\}.$$

As a result, since  $0 \leq 1 - Z_- - f_m \leq 1$ , we obtain

$$\sqrt{(f - 1)^2 \star \mu} \leq \sqrt{(f^{\mathbb{F}} - 1)^2 \frac{1 - Z_- - f_m}{(1 - Z_-)^2} I_{\{Z_- < 1\}} \star \mu} + \sqrt{\frac{f_m^2}{(1 - Z_-)^2} I_{\{Z_- < 1\}} \star \mu}.$$

Thus, a combination of this with the local boundedness of  $(1 - Z_-)^{-2} I_{\{Z_- < 1\}}$  (see Lemma 2.6), (4.10), and  $f_m^2 \star \mu \in \mathcal{A}_{loc}^+(\mathbb{F})$  (see Lemma 4.1), the proof of  $\sqrt{(f - 1)^2 \star \mu^{(1)}} = \sqrt{(f - 1)^2 I_{\{\psi > 0 \text{ \& } Z_- < 1\}} \star \mu} \in \mathcal{A}_{loc}^+(\mathbb{F})$  is completed. This ends the proof of the theorem.  $\square$

## 4.3 Proof of Theorem 2.18

In virtue of (4.7) and  $\llbracket \tau, +\infty \rrbracket \subset \{Z_- < 1\}$ , the assumption (2.13) holds iff

$$\begin{aligned} 0 &= E \left[ (1 - Z_-)^{-1} I_{\{Z_- + f_m = 1 > Z_-\}} I_{\llbracket \tau, +\infty \rrbracket} \star \nu_\infty \right] \\ &= E \left[ I_{\{Z_- + f_m = 1 > Z_-\}} \star \nu_\infty \right] = E \left[ I_{\{Z_- + f_m = 1 > Z_-\}} \star \mu_\infty \right]. \end{aligned}$$

This implies that  $I_{\{Z_- + f_m = 1 > Z_-\}} \star \nu$  and  $I_{\{Z_- + f_m = 1 > Z_-\}} \star \mu$  are null. Thus, we deduce that  $m^{(1)} = I_{\{Z_- + f_m = 1 > Z_-\}} \star \mu - f_m I_{\{Z_- + f_m = 1 > Z_-\}} \star \nu$  is also null. Then, the proof of the theorem follows immediately from combining this with Proposition 4.5 and Corollary 2.16–(ii).  $\square$

## APPENDIX

## A Deflators via Predictable Characteristics

Most results of this section are elaborated in [1], and we refer the reader to the appendix of that paper for details. Herein, we consider given a probability  $Q$ , a filtration  $\mathbb{H}$ , and a  $(\mathbb{H}, Q)$ -quasi-left-continuous semimartingale  $X$ . To this process, we associate the random measure of its jumps, denoted by  $\mu_X$ , and its  $(\mathbb{H}, Q)$ -compensator is denoted by  $\nu_X$ . We suppose that  $X$  has the following conical decomposition

$$X = X_0 + X^c + h \star (\mu_X - \nu_X) + (x - h) \star \mu_X + b \cdot A.$$

Here  $h(x) := x I_{\{|x| \leq 1\}}$  and  $h \star (\mu_X - \nu_X)$  represents the unique pure jumps  $(\mathbb{H}, Q)$ -local martingale with jumps taking the form of  $h(\Delta S) I_{\{\Delta S \neq 0\}}$ . We suppose that  $\nu_X(dt, dx) = F(t, dx) dA_t$ , and  $c$  the matrix such that  $\langle X^c \rangle = c \cdot A$ . The quadruplet  $(b, c, F, A)$  is the predictable characteristics of  $X$  under  $(\mathbb{H}, Q)$ . Here the elements of this quadruplet depends on  $(X, Q, \mathbb{H})$ , but there is no risk of confusion in this part.

**Theorem A.1** *Let  $(X, Q, \mathbb{H})$  be a quasi-left-continuous model, and  $(b^Q, c, F^Q, A)$  be its predictable characteristics under  $(\mathbb{H}, Q)$ . Then,  $X$  satisfies the NUPBR $(\mathbb{H}, Q)$  if and only if there exists a pair  $(\beta, f)$ , of  $\mathbb{H}$ -predictable process  $\beta$  and  $\tilde{\mathcal{P}}(\mathbb{H})$ -measurable functional  $f$ , such that*

$$f > 0, \quad \beta^{tr} c \beta \cdot A + \sqrt{(f - 1)^2 \star \mu_X} \in \mathcal{A}_{loc}^+(\mathbb{H}, Q), \quad (\text{A.1})$$

$$\int |x f(x) - h(x)| F(dx) < +\infty, \quad Q \otimes A - a.e. \quad (\text{A.2})$$

$$b + c\beta + \int [x f(x) - h(x)] F(dx) = 0, \quad Q \otimes A - a.e. \quad (\text{A.3})$$

See [1] for the proof.

**Proposition A.2** *Let  $X$  be an  $\mathbb{H}$ -adapted process. Then, the following assertions are equivalent.*

- (a) *There exists a sequence  $(T_n)_{n \geq 1}$  of  $\mathbb{H}$ -stopping times that increases to  $+\infty$ , such that for each  $n \geq 1$ , there exists a probability  $Q_n$  on  $(\Omega, \mathcal{H}_{T_n})$  such that  $Q_n \sim P$  and  $X^{T_n}$  satisfies  $NUPBR(\mathbb{H})$  under  $Q_n$ .*
- (b)  *$X$  satisfies  $NUPBR(\mathbb{H})$ .*
- (c) *There exists an  $\mathbb{H}$ -predictable process  $\phi$ , such that  $0 < \phi \leq 1$  and  $(\phi \cdot X)$  satisfies  $NUPBR(\mathbb{H})$ .*

The proof of this proposition can be found in Aksamit et al. [1].

**Proposition A.3** *Suppose that  $\tau$  is a honest time, and let  $H^{\mathbb{G}}$  be an  $\tilde{\mathcal{P}}(\mathbb{G})$ -measurable functional. Then, the following assertions hold.*

- (a) *There exist two  $\tilde{\mathcal{P}}(\mathbb{F})$ -measurable functional  $H^{\mathbb{F}}$  and  $K^{\mathbb{F}}$  such that*

$$H^{\mathbb{G}}(\omega, t, x) = H^{\mathbb{F}}(\omega, t, x)I_{\llbracket 0, \tau \rrbracket} + K^{\mathbb{F}}(\omega, t, x)I_{\llbracket \tau, +\infty \rrbracket}. \quad (\text{A.4})$$

- (b) *If furthermore  $H^{\mathbb{G}} > 0$  (respectively  $H^{\mathbb{G}} \leq 1$ ), then we can choose  $K^{\mathbb{F}} > 0$  (respectively  $K^{\mathbb{F}} \leq 1$ ) in (A.4).*

*Proof* The proofs of assertions (a) and (b) follow from mimicking Jeulin's proof [19, Proposition 5,3], and will be omitted herein.

## B $\mathbb{G}$ -local Integrability versus $\mathbb{F}$ -local Integrability

This subsection connects the  $\mathbb{G}$ -localisation and the  $\mathbb{F}$ -localisation for the part after  $\tau$ . This completes the analysis of [1] regarding the issue of local integrability under  $\mathbb{F}$  and  $\mathbb{G}$ , where the part up to  $\tau$  is fully discussed. There is a major difference between the current results and those of [1], which lies in the fact that for the case up to  $\tau$  we loose information after an  $\mathbb{F}$ -stopping when we pass to  $\mathbb{F}$ . However, for the part after  $\tau$ , as long as  $\tau$  is finite, we pass from  $\mathbb{G}$ -localisation to  $\mathbb{F}$ -localisation without any loss of information. The following is the most innovative result of the appendix.

**Proposition B.1** *The following assertions hold.*

- (a) *If  $\tau$  is a finite almost surely honest time and  $(\sigma_n^{\mathbb{G}})_{n \geq 1}$  is a sequence of finite  $\mathbb{G}$ -stopping times that increases to infinity, then there exists a sequence of finite  $\mathbb{F}$ -stopping times,  $(\sigma_n^{\mathbb{F}})_{n \geq 1}$ , that increases to infinity as well and*

$$\max(\sigma_n^{\mathbb{G}}, \tau) = \max(\sigma_n^{\mathbb{F}}, \tau), \quad P - a.s. \quad (\text{B.1})$$

- (b) *If  $\tau \in \mathcal{H}$  and is finite almost surely, then there exists a sequence of  $\mathbb{F}$ -stopping times,  $(\sigma_n)_{n \geq 1}$ , that increases to infinity almost surely and*

$$\left\{ Z_- < 1 \right\} \cap \llbracket 0, \sigma_n \rrbracket \subset \left\{ 1 - Z_- \geq \frac{1}{n} \right\}, \quad \forall n \geq 1. \quad (\text{B.2})$$

*Or equivalently,  $(1 - Z_-)^{-1}I_{\{Z_- < 1\}}$  is  $\mathbb{F}$ -locally bounded when  $\tau \in \mathcal{H}$  and is finite almost surely.*



*Proof* The proof of this proposition is given in two parts where we prove assertions (a) and (b) respectively.

1) The proof of assertion (a) boils down to the following fact:

for any  $\mathbb{G}$ -stopping time,  $\sigma^{\mathbb{G}}$ , there exists an  $\mathbb{F}$ -stopping time,  $\sigma^{\mathbb{F}}$  such that

$$\sigma^{\mathbb{G}} \vee \tau = \sigma^{\mathbb{F}} \vee \tau \quad P - a.s. \quad (\text{B.3})$$

Indeed, if this fact holds, then there exists  $\mathbb{F}$ -stopping times,  $(\sigma_n)_{n \geq 1}$  such that for any  $n \geq 1$ , the pair  $(\sigma_n^{\mathbb{G}}, \sigma_n)$  satisfies (B.1). Since  $\sigma_n^{\mathbb{G}}$  increases with  $n$ , by putting  $\sigma_n^{\mathbb{F}} := \sup_{1 \leq k \leq n} \sigma_k$ , we can easily prove that the pair  $(\sigma_n^{\mathbb{G}}, \sigma_n^{\mathbb{F}})$  satisfies (B.1) as well (this is due to  $\max_{1 \leq i \leq n} (x_i \vee y) = (\max_{1 \leq i \leq n} x_i) \vee y$  for any nonnegative  $x_i, y$ ). Then, assertion (a) follows immediately from taking the limit in (B.1) and making use of  $\tau < +\infty$  P-a.s. which implies that  $\sup_{n \geq 1} \sigma_n = \lim_{n \rightarrow +\infty} \sigma_n^{\mathbb{F}} = +\infty$  P-a.s. This shows that the proof of assertion (a) is achieved as long as we prove the claim (B.3). This is the main focus of the remaining part of this proof.

By applying the proposition below (which is fully due to Barlow [6]) to the process  $Y^{\mathbb{G}} = I_{[\sigma^{\mathbb{G}} \vee \tau, +\infty[}$ , we obtain the existence of an  $\mathbb{F}$ -progressively measurable process  $K^{\mathbb{F}}$  such that

$$Y^{\mathbb{G}} = K^{\mathbb{F}} I_{[\tau, +\infty[}. \quad (\text{B.4})$$

Thus, it is easy that one can replace  $K^{\mathbb{F}}$  with  $I_{\{K^{\mathbb{F}}=1\}}$ . Since  $[\tau, +\infty[ \subset \{Z < 1\}$ , It is also easy to check that one can choose  $K^{\mathbb{F}}$  such that, on  $\{\tau < \sigma^{\mathbb{G}}, \{K^{\mathbb{F}} = 1\} \subset \{Z < 1\}$ . Then, put

$$\sigma := \inf\{t \geq 0 : K_t^{\mathbb{F}} = 1\}. \quad (\text{B.5})$$

This is an  $\mathbb{F}$ -stopping time, and due to  $[\sigma^{\mathbb{G}} \vee \tau, +\infty[ \subset \{K^{\mathbb{F}} = 1\}$ , we get

$$\sigma \leq \tau \vee \sigma^{\mathbb{G}} \quad P - a.s. \quad (\text{B.6})$$

By applying Proposition B.2, we deduce the existence of two double sequence of  $\mathbb{F}$ -stopping times  $(\alpha_{nm})_{n,m \geq 1}$  and  $(\beta_{nm})_{n,m \geq 1}$  satisfying the four assertions of the proposition. As a result, we get, on  $\{\tau < \sigma^{\mathbb{G}}, \{K^{\mathbb{F}} = 1\} \subset \{Z < 1\}$ , we have

$$[\sigma^{\mathbb{G}}, +\infty[ \subset \{K^{\mathbb{F}} = 1\} \subset \bigcup_{n,m \geq 1} [\alpha_{nm}, \beta_{nm}[.$$

By combining this with (B.5), we deduce that

$$\{\tau < \sigma^{\mathbb{G}}\} \subset \bigcup_{n,m \geq 1} \{\alpha_{nm} \leq \sigma \leq \sigma^{\mathbb{G}} < \beta_{nm}\}.$$

Thanks to assertions (i) and (ii) of Proposition B.2, that claims that  $\tau$  takes values in  $[\beta_{n(m-1)}, \alpha_{nm}[$  only, we get  $\{\alpha_{nm} \leq \sigma \leq \sigma^{\mathbb{G}} < \beta_{nm}\} = \{\tau < \alpha_{nm} \leq \sigma \leq \sigma^{\mathbb{G}} < \beta_{nm}\}$ , and hence

$$\{\tau < \sigma^{\mathbb{G}}\} \subset \bigcup_{n,m \geq 1} \{\tau < \alpha_{nm} \leq \sigma \leq \sigma^{\mathbb{G}} < \beta_{nm}\}. \quad (\text{B.7})$$

Now, due to (B.4) and the fact that on  $[\sigma, \sigma + \epsilon[\cap\{K^{\mathbb{F}} = 1\} \neq \emptyset$   $P$ -a.s, we deduce that  $(\tau \leq \sigma < \sigma^{\mathbb{G}})$  is an impossible event. Therefore, (B.7) leads to

$$\{\tau < \sigma^{\mathbb{G}}\} \subset \{\sigma = \sigma^{\mathbb{G}}\}.$$

Hence, by combing this with (B.6), we derive

$$\begin{aligned} \tau \vee \sigma^{\mathbb{G}} &= (\tau \vee \sigma^{\mathbb{G}})I_{\{\sigma^{\mathbb{G}} \leq \tau\}} + (\tau \vee \sigma^{\mathbb{G}})I_{\{\tau < \sigma^{\mathbb{G}}\}} = \tau I_{\{\sigma^{\mathbb{G}} \leq \tau\}} + (\tau \vee \sigma)I_{\{\tau < \sigma^{\mathbb{G}}\}} \\ &= (\tau \vee \sigma)I_{\{\sigma^{\mathbb{G}} \leq \tau\}} + (\tau \vee \sigma)I_{\{\tau < \sigma^{\mathbb{G}}\}} = \tau \vee \sigma. \end{aligned}$$

This proves (B.3), and the prof of assertion (a) is completed.

2) Here, we prove assertion (b). Since  $\tau \in \mathcal{H}$ , then  $(1 - Z_-)^{-1}I_{\llbracket \tau, +\infty \rrbracket}$  is  $\mathbb{G}$ -locally bounded due to Lemma 2.6-(b). Thus, on the one hand, there exists a sequence of  $\mathbb{G}$ -stopping times,  $(\sigma_n^{\mathbb{G}})_{n \geq 1}$  that increases to infinity almost surely and

$$\llbracket \tau, +\infty \rrbracket \cap \llbracket 0, \sigma_n^{\mathbb{G}} \rrbracket \subset \left\{1 - Z_- \geq 1/n\right\}. \quad (\text{B.8})$$

On the other hand, thanks to assertion (a), there exists a sequence of  $\mathbb{F}$ -stopping times,  $(\sigma_n)_{n \geq 1}$  that increases to infinity almost surely and satisfies (B.1). Then, by inserting this in (B.8), we get

$$\llbracket \tau, +\infty \rrbracket \cap \llbracket 0, \sigma_n \rrbracket \subset \left\{1 - Z_- \geq 1/n\right\}.$$

By taking the  $\mathbb{F}$ -predictable projection on both side, we get

$$0 \leq (1 - Z_-)I_{\llbracket 0, \sigma_n \rrbracket} \leq I_{\left\{1 - Z_- \geq 1/n\right\}}.$$

This implies that  $\left\{1 - Z_- < 1/n\right\} \subset \{Z_- = 1\} \cup \llbracket \sigma_n, +\infty \rrbracket$ , which is equivalent to (B.2). Hence, the proof of assertion (b) is achieved and that of the proposition as well.  $\square$

**Proposition B.2** *Suppose that  $\tau$  is a honest time. Then, the following hold.*

(i) *There exists two double sequences of  $\mathbb{F}$ -stopping times  $(\alpha_{n,m})_{n,m \geq 1}$  and  $(\beta_{n,m})_{n,m \geq 1}$  such that  $\alpha_{n,m} \leq \beta_{n,m}$   $P$ -a.s. for all  $n, m \geq 1$ , and*

$$\llbracket \tau, +\infty \rrbracket \subset \{Z < 1\} \subset \bigcup_{n,m \geq 1} \llbracket \alpha_{n,m}, \beta_{n,m} \rrbracket. \quad (\text{B.9})$$

(ii) *For any  $n, m \geq 1$ ,  $\{\tau \geq \alpha_{nm}\} \subset \{\tau \geq \beta_{nm}\}$   $P$ -a.s.*

(iii) *For any  $\mathbb{G}$ -optional process  $Y^{\mathbb{G}}$ , there exists an  $\mathbb{F}$ -progressively measurable process  $K^{\mathbb{F}}$  such that*

$$Y^{\mathbb{G}}I_{\llbracket \tau, +\infty \rrbracket} = K^{\mathbb{F}}I_{\llbracket \tau, +\infty \rrbracket}. \quad (\text{B.10})$$

(iv) *For any  $\mathbb{G}$ -optional càdlàg process  $Y^{\mathbb{G}}$  such that  $Y^{\mathbb{G}} = 0$  on  $\llbracket 0, \alpha_{n,m} \rrbracket$  and constant on  $\llbracket \beta_{n,m}, +\infty \rrbracket$ , there exists an  $\mathbb{F}$ -progressively measurable process  $K^{\mathbb{F}}$  that is càdlàg and satisfies (B.10).*

*Proof* For the proof we refer the reader to [6]. In fact, assertion (i) is exactly Lemma 4.1-(iv) in [6], while the assertion (ii) is a combination of Proposition 4.3 and Lemma 4.4-(ii) of the same paper.

The next result addresses the  $\mathbb{G}$ -local integrability involving the random measures that is vital for the proof of Theorem 2.14.

**Proposition B.3** *Suppose that  $\tau \in \mathcal{H}$  is finite almost surely. Let  $\Phi_\alpha(\cdot)$  (for  $\alpha > 0$ ) be defined in (4.9). Then, the following properties hold.*

- (a) *Let  $f$  be a real-valued and  $\tilde{\mathcal{P}}(\mathbb{H})$ -measurable functional. Then,  $\sqrt{(f-1)^2} \star \mu$  belongs to  $\mathcal{A}_{loc}^+(\mathbb{H})$  if and only if  $\Phi_\alpha(f) \star \mu \in \mathcal{A}_{loc}^+(\mathbb{H})$  does.*
- (b) *Let  $f$  be a real-valued and  $\tilde{\mathcal{P}}(\mathbb{H})$ -measurable functional. Then,  $\sqrt{(f-1)^2} I_{\tau, +\infty} \star \mu \in \mathcal{A}_{loc}^+(\mathbb{G})$  iff  $\Phi_\alpha(f)(1 - Z_- - f_m) I_{\{Z_- < 1\}} \star \mu \in \mathcal{A}_{loc}^+(\mathbb{F})$ .*
- (c) *Let  $\phi$  be nonnegative and  $\mathbb{F}$ -predictable process. Then,  $P \otimes A$ -a.e.  $I_{\tau, +\infty} \llcorner \{\phi < +\infty\}$  if and only if  $\{Z_- < 1\} \subset \{\phi < +\infty\}$ .*
- (d) *Let  $\phi$  be an  $\mathbb{F}$ -predictable process. Then,  $P \otimes A$ -a.e.  $I_{\tau, +\infty} \llcorner \{\phi = 0\}$  if and only if  $\{Z_- < 1\} \subset \{\phi = 0\}$ .*

*Proof* (a) Assertion (a) is borrowed from [1] ( see Proposition C.3-(a)).

(b) Thanks to assertion (a), we deduce that  $\sqrt{(f-1)^2} I_{\tau, +\infty} \star \mu \in \mathcal{A}_{loc}^+(\mathbb{G})$  iff  $\Phi_\alpha(f) \star \mu^{\mathbb{G}} \in \mathcal{A}_{loc}^+(\mathbb{G})$  iff

$$\Phi_\alpha(f) \left( 1 - \frac{f_m}{1 - Z_-} \right) I_{\tau, +\infty} \star \nu = \Phi_\alpha(f) \star \nu^{\mathbb{G}} \in \mathcal{A}_{loc}^+(\mathbb{G}). \quad (\text{B.11})$$

Then, a direct application of Lemma 3.2-(d) to the pair

$$(\varphi, V) := \left( [1 - f_m(1 - Z_-)^{-1}] I_{\{Z_- < 1\}}, \Phi_\alpha(f) \star \nu \right),$$

the proof of assertion (b) follows immediately.

(c) Suppose that  $P \otimes A$ -a.e. that  $I_{\tau, +\infty} \llcorner \{\phi < +\infty\}$ . This is equivalent to  $I_{\tau, +\infty} \llcorner \leq I_{\{\phi < +\infty\}}$   $P \otimes A$ -a.e.. Then, by taking the  $\mathbb{F}$ -predictable projection on both sides, we get  $1 - Z_- \leq I_{\{\phi < +\infty\}}$   $P \otimes A$ -a.e.. This obviously proves that  $I_{\tau, +\infty} \llcorner \{\phi < +\infty\}$  implies  $\{Z_- < 1\} \subset \{\phi < +\infty\}$ . The reverse sense follows from  $I_{\tau, +\infty} \llcorner \{Z_- < 1\}$ . This ends the proof of assertion (c).

(d) The proof of assertion (d) mimics the proof of assertion (c), and will be omitted. This ends the proof of the proposition.  $\square$

## C Proofs for Lemmas 3.1 and 3.2 of Subsection 3.1

*Proof of Lemma 3.1* The proof of the lemma will be achieved in three steps.

1) This step proves assertion (a). From Lemma 2.6

$$I_{\tau, +\infty} \llcorner \cdot V - I_{\tau, +\infty} \llcorner \cdot V^{p, \mathbb{F}} + I_{\tau, +\infty} \llcorner (1 - Z_-)^{-1} \cdot \langle V, m \rangle^{\mathbb{F}}$$

is a  $\mathbb{G}$ -local martingale, hence

$$\begin{aligned} (I_{\llbracket \tau, +\infty \rrbracket} \cdot V)^{p, \mathbb{G}} &= I_{\llbracket \tau, +\infty \rrbracket} \cdot V^{p, \mathbb{F}} - I_{\llbracket \tau, +\infty \rrbracket} (1 - Z_-)^{-1} \cdot \langle V, m \rangle^{\mathbb{F}} \\ &= I_{\llbracket \tau, +\infty \rrbracket} \cdot V^{p, \mathbb{F}} - I_{\llbracket \tau, +\infty \rrbracket} (1 - Z_-)^{-1} \cdot (\Delta m \cdot V)^{p, \mathbb{F}} \\ &= I_{\llbracket \tau, +\infty \rrbracket} (1 - Z_-)^{-1} \cdot \left( (1 - Z_- - \Delta m) \cdot V \right)^{p, \mathbb{F}}, \end{aligned}$$

where the second equality follows from Yoeurp's lemma. This ends the proof of (3.1). The equality (3.2) follows immediately from (3.1) by taking the jumps in both sides, and using  $\Delta(K^{p, \mathbb{H}}) = {}^{p, \mathbb{H}}(\Delta K)$  when both terms exist.

**2)** Now, we prove assertion (b). By applying (3.2) for  $V_{\epsilon, \delta} \in \mathcal{A}_{loc}(\mathbb{F})$  given by

$$V_{\epsilon, \delta} := \sum (\Delta M)(1 - \tilde{Z})^{-1} I_{\{|\Delta M| \geq \epsilon, 1 - \tilde{Z} \geq \delta\}},$$

we get, on  $\llbracket \tau, +\infty \rrbracket$ ,

$${}^{p, \mathbb{G}} \left( (\Delta M)(1 - \tilde{Z})^{-1} I_{\{|\Delta M| \geq \epsilon, 1 - \tilde{Z} \geq \delta\}} \right) = (1 - Z_-)^{-1} {}^{p, \mathbb{F}} \left( \Delta M I_{\{|\Delta M| \geq \epsilon, 1 - \tilde{Z} \geq \delta\}} \right).$$

Then, the first equality in (3.3) follows from letting  $\epsilon$  and  $\delta$  go to zero, and we get on  $\llbracket \tau, +\infty \rrbracket$

$${}^{p, \mathbb{G}} \left( \frac{\Delta M}{1 - \tilde{Z}} \right) = (1 - Z_-)^{-1} {}^{p, \mathbb{F}} \left( \Delta M I_{\{1 - \tilde{Z} > 0\}} \right) = (1 - Z_-)^{-1} {}^{p, \mathbb{F}} \left( \Delta M I_{\{\tilde{Z} < 1\}} \right).$$

To prove the second equality in (3.3), we write that, on  $\llbracket \tau, +\infty \rrbracket$ ,

$$\begin{aligned} {}^{p, \mathbb{G}} \left( \frac{1}{1 - \tilde{Z}} \right) &= (1 - Z_-)^{-1} + (1 - Z_-)^{-1} {}^{p, \mathbb{G}} \left( \frac{\Delta m}{1 - \tilde{Z}} \right) \\ &= (1 - Z_-)^{-1} + (1 - Z_-)^{-2} {}^{p, \mathbb{F}} \left( \Delta m I_{\{1 - \tilde{Z} > 0\}} \right) \\ &= (1 - Z_-)^{-1} - (1 - Z_-)^{-1} {}^{p, \mathbb{F}} \left( I_{\{\tilde{Z} = 1\}} \right) = (1 - Z_-)^{-1} {}^{p, \mathbb{F}} \left( I_{\{\tilde{Z} < 1\}} \right). \end{aligned}$$

The second equality is due to (3.2), and the third equality follows from combining  ${}^{p, \mathbb{F}}(\Delta m) = 0$ , and  $\Delta m = \tilde{Z} - Z_-$ . This ends the proof of assertion (b).

**3)** The proof of (3.4) follows immediately from assertion (b) and the fact that the thin process  ${}^{p, \mathbb{F}} \left( \Delta M I_{\{\tilde{Z} < 1\}} \right)$  may take nonzero values on countably many predictable stopping times only, on which  $\Delta M$  already vanishes. This completes the proof of the lemma.  $\square$

*Proof Lemma 3.2* The proof of the lemma is given in three parts. In the first part we prove both assertions (a) and (b), while in the second and the third parts we focus on assertions (c) and (d) respectively.

1) Let  $V$  be an  $\mathbb{F}$ -adapted process with finite variation. Then, we obtain

$$\text{Var}(U) = (1 - Z)^{-1} I_{\llbracket \tau, +\infty \rrbracket} \cdot \text{Var}(V).$$

Therefore, since  $1 - \tilde{Z}_t = P(\tau < t | \mathcal{F}_t) \leq 1 - Z_t$ , for any bounded and  $\mathbb{F}$ -optional process  $\phi$  such that  $\phi \cdot \text{Var}(V) \in \mathcal{A}^+(\mathbb{F})$ , we obtain

$$\begin{aligned} E\left[(\phi \cdot \text{Var}(U))_\infty\right] &= E\left(\int_0^\infty \frac{\phi_t I_{\{t > \tau\}}}{1 - Z_t} d\text{Var}(V)_t\right) \\ &= E\left(\int_0^\infty \frac{\phi_t P(\tau < t | \mathcal{F}_t)}{1 - Z_t} I_{\{Z_t < 1\}} d\text{Var}(V)_t\right) \leq E\left[(\phi \cdot \text{Var}(V))_\infty\right]. \end{aligned} \quad (\text{C.1})$$

As a result, by taking  $\phi = I_{\llbracket 0, \sigma \rrbracket}$  in (C.1), for an  $\mathbb{F}$ -stopping time  $\sigma$  such that  $\text{Var}(V)^{\sigma-} \in \mathcal{A}^+(\mathbb{F})$ , we get  $E[\text{Var}(U)_{\sigma-}] \leq E[\text{Var}(V)_{\sigma-}]$ . This proves that the process  $U$  has a finite variation and hence is well defined as well. Being  $\mathbb{G}$ -adapted for  $U$  is obvious, while being càdlàg follows immediately from (C.1). This ends the proof of assertion (a).

To prove assertion (b), we assume that  $V \in \mathcal{A}_{loc}(\mathbb{F})$  and consider  $(\vartheta_n)_{n \geq 1}$ , a sequence of  $\mathbb{F}$ -stopping times that increases to  $+\infty$  such that  $\text{Var}(V)^{\vartheta_n} \in \mathcal{A}^+(\mathbb{F})$ . Then, by choosing  $\phi = I_{\llbracket 0, \vartheta_n \rrbracket}$  in (C.1), we conclude that  $U$  belongs to  $\mathcal{A}_{loc}(\mathbb{G})$  whenever  $V$  does under  $\mathbb{F}$ . For the case when  $V \in \mathcal{A}(\mathbb{G})$ , it is enough to take  $\phi = 1$  in (C.1), and conclude that  $U \in \mathcal{A}(\mathbb{G})$ . To prove (3.6), for any  $n \geq 1$ , we put

$$U_n := (1 - Z)^{-1} I_{\llbracket \tau, +\infty \rrbracket} I_{\{\tilde{Z} \leq 1 - \frac{1}{n}\}} \cdot V = \left(1 - \tilde{Z}\right)^{-1} I_{\llbracket \tau, +\infty \rrbracket} I_{\{\tilde{Z} \leq 1 - \frac{1}{n}\}} \cdot V, \quad n \geq 1.$$

Then, thanks to (3.1), we derive

$$U^{p, \mathbb{G}} = \lim_{n \rightarrow +\infty} (U_n)^{p, \mathbb{G}} = \lim_{n \rightarrow +\infty} (1 - Z_-)^{-1} I_{\llbracket \tau, +\infty \rrbracket} \cdot \left(I_{\{\tilde{Z} \leq 1 - \frac{1}{n}\}} \cdot V\right)^{p, \mathbb{F}}.$$

This clearly implies (3.6).

2) It is easy to see that it is enough to prove the assertion for the case when  $V$  is nondecreasing. Thus, suppose that  $V$  is nondecreasing. It obvious that  $(1 - \tilde{Z}) \cdot V \in \mathcal{A}_{loc}^+(\mathbb{F})$  implies  $I_{\llbracket \tau, +\infty \rrbracket} \cdot V \in \mathcal{A}_{loc}^+(\mathbb{G})$ . Hence, for the rest of this part, we focus on proving the reverse. Suppose  $I_{\llbracket \tau, +\infty \rrbracket} \cdot V \in \mathcal{A}_{loc}^+(\mathbb{G})$ . Then, there exists a sequence  $\mathbb{G}$ -stopping times that increases to infinity and  $(I_{\llbracket \tau, +\infty \rrbracket} \cdot V)^{\sigma_n^{\mathbb{G}}} \in \mathcal{A}^+(\mathbb{G})$ . Thanks to Proposition A.3-(c), we obtain a sequence of  $\mathbb{F}$ -stopping times,  $(\sigma_n^{\mathbb{F}})_{n \geq 1}$ , that increases to infinity and  $\sigma_n^{\mathbb{G}} \vee \tau = \tau \vee \sigma_n^{\mathbb{F}}$ . Therefore, we get  $(I_{\llbracket \tau, +\infty \rrbracket} \cdot V)^{\sigma_n^{\mathbb{G}}} \equiv (I_{\llbracket \tau, +\infty \rrbracket} \cdot V)^{\sigma_n^{\mathbb{F}}}$  and hence

$$E\left((1 - \tilde{Z}) \cdot V_{\sigma_n^{\mathbb{F}}}\right) = E\left(I_{\llbracket \tau, +\infty \rrbracket} \cdot V_{\sigma_n^{\mathbb{F}}}\right) = E\left(I_{\llbracket \tau, +\infty \rrbracket} \cdot V_{\sigma_n^{\mathbb{G}}}\right) < +\infty. \quad (\text{C.2})$$

This proves that the process  $(1 - \tilde{Z}) \cdot V$  belongs to  $\mathcal{A}_{loc}^+(\mathbb{F})$ , and the proof of assertion (c) is achieved.

3) The proof of assertion (d) follows all the steps of the proof of assertion (c), except (C.2) which takes the form of

$$E((1 - Z_-)\varphi \cdot V_{\sigma_n^{\mathbb{F}}}) = E(I_{\tau, +\infty} \varphi \cdot V_{\sigma_n^{\mathbb{F}}}) = E(I_{\tau, +\infty} \varphi \cdot V_{\sigma_n^{\mathbb{G}}}) < +\infty$$

instead due to the predictability of  $V$ . This proves that  $I_{\tau, +\infty} \varphi \cdot V \in \mathcal{A}_{loc}^+(\mathbb{G})$  if and only if  $(1 - Z_-)\varphi \cdot V \in \mathcal{A}_{loc}^+(\mathbb{F})$ , while the equivalence  $(1 - Z_-)\varphi \cdot V \in \mathcal{A}_{loc}^+(\mathbb{F})$  iff  $I_{\{Z_- < 1\}} \varphi \cdot V \in \mathcal{A}_{loc}^+(\mathbb{F})$  follows from the fact that  $(1 - Z_-)^{-1} I_{\{Z_- < 1\}}$  is  $\mathbb{F}$ -locally bounded (see Proposition B.1-(b)). This ends the proof of assertion (d) and the proof of the lemma as well.  $\square$

## D Proof of Proposition 4.5

The proof relies essentially on an adequate application(s) of Theorem A.1. To this end, we consider

$$Z^{(\psi)} := \mathcal{E}(N^{(\psi)}) \quad \text{where} \quad N^{(\psi)} := (\psi - 1)I_{\{\psi > 0\}} \star (\mu - \nu),$$

and remark that  $Z^{(\psi)}$  is a positive  $\mathbb{F}$ -local martingale. Hence, in virtue of Proposition A.2, we can assume without loss of generality that  $Z^{(\psi)}$  is a uniformly integrable martingale, and put  $Q := Z_{\infty}^{\psi} \cdot P$  (probability measure equivalent to  $P$ ). Thus, the rest of the proof applies Theorem A.1 to both models  $(S^{(1)}, Q, \mathbb{F})$  and  $(S^{(0)} := I_{\{Z_- < 1\}} \cdot \mathcal{T}_a(S), \mathbb{F})$ , and compare the conditions (A.1)-(A.2)-(A.3) associated to these models. To this end, we need to derive the predictable characteristics,  $(b^{(1,Q)}, c^{(1,Q)}, F^{(1,Q)}(dx), A^{(1,Q)})$ , of  $(S^{(1)}, Q, \mathbb{F})$ . Then, the  $\mathbb{F}$ -compensator of  $\mu^{(1)}$  under  $Q$  is  $\nu^{(1,Q)}(dt, dx) := \psi(x)\nu(dt, dx)$  that coincides with  $\nu^{(0)}$ . Then, using  $(hI_{\{\tilde{Z}=1 > Z_-\}} \star \mu)^{p, \mathbb{F}} = h(1 - \psi)I_{\{Z_- < 1\}} \star \nu$  and that the compensator under  $Q$  of  $H \star \mu$ —for any nonnegative and  $\tilde{\mathcal{P}}(\mathbb{F})$ -measurable  $H$ —is  $H(I_{\{\psi=0\}} + \psi) \star \nu$ , we get

$$\begin{aligned} b^{(1,Q)} &:= b - \int h(x)(\psi(x) - 1)F(dx), & c^{(1,Q)} &:= c, \\ F^{(1,Q)}(dx) &:= \psi(x)F(dx), & A^{(1,Q)} &:= I_{\{Z_- < 1\}} \cdot A. \end{aligned}$$

By comparing the above quadruplet to the quadruplet given in (4.18), we conclude that the two models,  $(S^{(1)}, Q, \mathbb{F})$  and  $(S^{(0)} := I_{\{Z_- < 1\}} \cdot \mathcal{T}_a(S), \mathbb{F})$ , have the same predictable characteristics. Hence, the proof of the proposition follows immediately from applying Theorem A.1 and using the same pair of  $\mathbb{F}$ -predictable functionals (i.e.  $(\beta^{(0)}, f^{(0)}) = (\beta^{(1)}, f^{(1)})$ ), as the conditions (A.1)-(A.2)-(A.3) are exactly the same for both models.

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